# Preschool Geometry

## Theory, Research, and Practical Perspectives

Esther Levenson, Dina Tirosh and Pessia Tsamir


SensePublishers

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# FOREWORD

Recently the issue of early childhood mathematics has come to the fore and with it the importance of teaching geometrical concepts and reasoning from a young age. Research has not only demonstrated that young children can learn mathematics but that children's mathematics knowledge and reasoning should be actively promoted from an early age (Clements & Sarama, 2007). Specifically, geometry is not only in and of itself a key domain but it may also support the learning of other mathematical topics, such as number and patterns. Developing geometrical reasoning, progressing from visual to descriptive and analytical reasoning may go hand in hand with developing the ability to form well defined concepts in other domains as well. Unfortunately, young children with little mathematics knowledge tend to fall further behind their peers each year. Compounding this problem, early knowledge of mathematics is often seen as a predictor of later school success (Jimerson, Egelnad, & Teo, 1999).

With this in mind, it is not surprising to find increased calls for improving early childhood mathematics education, including the learning of geometrical concepts. At a recent 2009 Conference of European Research in Mathematics Education, a new working group in Early Years Mathematics was established in response to increased calls for research regarding mathematics learning and mathematics teacher education in the early years (ages 3-8). A joint position paper published in the United States by the National Association for the Education of Young Children (NAEYC) and the National Council for Teachers of Mathematics (NCTM) stated that "high quality, challenging, and accessible mathematics education for 3- to 6-year old children is a vital foundation for future mathematics learning" (NAEYC & NCTM, 2002, p. 1). Further evidence of concern for preschool mathematics education may be seen in the rise of national curricula in various countries which now make specific and sometimes mandatory recommendations for including mathematics and geometry as part of the preschool program. For example, in England, the Statutory Framework for the Early Years Foundation Stage (2008) states precise goals related to learning geometrical concepts during these years. In the US, the Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics (NCTM, 2006) specifically mention that children should be able to identify and describe a variety of two- and three-dimensional shapes presented in a variety of ways and use geometrical concepts when recognizing and working on simple sequential patterns or when analyzing a data set. Yet, geometry and spatial thinking are often ignored or minimized in early education (Sarama & Clements, 2009). Thus, there is an urgent need for the early childhood education community to improve geometry education in preschool.

This book is devoted entirely to the learning and teaching of geometry in preschool. The first part of the book is dedicated to children's geometrical thinking; the second part focuses on geometrical tasks; the third part focuses on teaching geometry to young children. Each of the three parts is structured in a similar manner, beginning with general theory and research, continuing with specific examples related to those theories, and moving on to elements of actual practice.

Part one is a study of preschool children's conceptualization of geometrical figures. As such, it begins with a review of theories and research related to concept formation in geometry. It then discusses more specifically the building of concept images in line with concept definitions, and how children's knowledge may be both assessed and promoted. It also discusses dilemmas that arise in the process. The second part of the book is devoted to geometrical tasks. It reviews the general structure and different elements of mathematical tasks and moves on to specifically discuss aspects of geometrical task design and implementation with young children. The second part also offers a review of several geometrical tasks implemented with young children and their role in developing and assessing geometrical reasoning. The third part of this book focuses on teaching geometry to young children. Taking into consideration that preschool children may attend a variety of day-care facilities or may be entirely home schooled, this part begins with theories and research related to the knowledge necessary for anyone who wishes to teach geometry to young children. It then continues with how this knowledge may be promoted, through, for example, professional development, and how this knowledge may then be put into practice. It also offers suggestions for tasks which may be implemented during professional development.

For whom did we write this book? First of all, we believe that this book will contribute greatly to preschool caregivers and teachers. Often, these practitioners receive little or no preparation for teaching mathematics to young children (Ginsburg, Lee, & Boyd, 2008). Yet, as we mentioned above, according to many national guidelines and curricula, they are responsible for teaching geometry in their classes. This book offers both a theoretical review as well as practical suggestions for how the teacher may promote geometrical learning in preschool. We also believe that this book will contribute to teacher educators, responsible for the professional development of both prospective and practicing preschool teachers. For the research community, each part of this book not only offers a review of previous research related to that section, but also raises many questions which point to the need for additional research. In general, any person who has an interest in the mathematics education of preschool children, be it parents, caregivers, formal, and informal educators, will find this book relevant. As you read this book, you may view it as an odyssey, an intellectual wandering and eventful journey, of learning and teaching geometry with preschool children. It is not a book to be read through in one sitting. It is a book to linger over, to take the time and contemplate the different examples and situations illustrated throughout. We hope that you will also find this book an eventful journey.

PART 1

# STUDYING PRESCHOOL CHILDREN'S DEVELOPMENT OF GEOMETRICAL CONCEPTS

This book is concerned with geometry in the preschool. In order to begin discussing how geometry might be introduced to young children and the kinds of tasks and activities which might promote geometrical thinking, it is necessary to first review how children develop geometrical thinking. The first chapter is dedicated to studying preschool children's development of geometrical concepts. We begin with an overview of theories related to how children acquire geometrical concepts and research concerned with developing geometrical thinking. We then focus on two-dimensional figures, examining separately the nuances and challenges associated with different shapes. Finally, we discuss three-dimensional figures.

The second and third chapters discuss how preschool children may come to build concept images in line with concept definitions.

CHAPTER 1

# THEORIES AND RESEARCH RELATED TO CONCEPT FORMATION IN GEOMETRY

In order for us to discuss with you, the reader, how geometrical concepts are developed, we need to establish a common language and a common background. This chapter provides terminology and theories on which the other sections and chapters of this book rest. It begins by presenting theories related to concept formation in general, proceeds to theories related to concept formation in mathematics, and finally discusses concept formation in geometry. But first, what do we mean when we refer to a ‘concept’? “Cognition does not start with concepts, but the other way around: concepts are the *result* of cognitive processes” (Freudenthal, 1991, p. 18). Concepts arise from the manipulation of mental objects. It may be seen as the end-product of becoming aware of similarities among our experiences and classifying these experiences based on their similarities. In other words, it is the end-product of abstraction (Skemp, 1971).

## 1.1 THE NATURE OF CONCEPTS

How are concepts formed within the mind of a person? Take, for example, the concepts of ‘dog’ and ‘cat’. Both a dog and a cat are four-legged animals. So how does a child learn to differentiate between them?

Concept formation is related to categorization. Think about a bird. Do you have a picture in your mind? Now think of another example of a bird. Can you think of yet another example? Within cognitive psychology, several theories attempt to describe processes of categorization and of concept formation. Two major theories are the classical view and the probabilistic (or prototype) view. According to the classical view, concepts and categories are represented by a set of defining features. For example, birds have defining features such as being bipeds and having wings. Instances of a concept, also called exemplars or examples, share common properties that are necessary and sufficient conditions for defining the concept (Klausmeier & Sipple, 1980; Smith, Shoben, & Rips, 1974; Smith & Medin, 1981). The features of a new stimulus would then be judged against the features of a known category in order to determine if it is an example of that category. What examples of birds did you come up with? Did you think of a chicken? Is a chicken a bird? Is it a biped and does it have wings? Yes. It is a biped and it does have wings. Therefore, a chicken is a bird. But it doesn’t perch in trees, you may exclaim. Perching in trees might be considered a characteristic feature but not a defining feature. In other words, some birds may perch in trees but it is not necessary for the chicken to perch in trees in order for it to be an example of a bird.

The classical view assumes clear-cut boundaries by which category membership can be determined. But this is not always the case. For example, is your living room carpet part of the furniture of that room? Some might answer yes and others might answer no. On the one hand, it may be considered part of the decorative furnishings of the living room. On the other hand, it is not intended to sit on, hold objects, or store things. The probabilistic view takes into account characteristic features and not just defining features. In other words, if an example has enough characteristic features, or if the characteristic features it has are the more acceptable and known features, then it can still be considered an example of that concept.

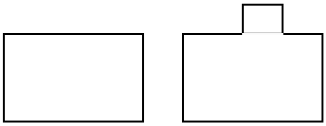
Because concepts are represented by a set of features which are characteristic or probable of examples, members of a category may be graded, with some instances considered to be "better" examples than others. Think back again to the examples of birds which came to your mind previously. You probably did not think of a chicken although we already established that a chicken is technically a bird. Does your bird have a particular color? A typical size? The features of that bird you envision are not defining. They are characteristic. The probabilistic theory also proposes the existence of ideal examples, called prototypes, which are often acquired first and serve as a basis for comparison when categorizing additional examples and nonexamples (Attneave, 1957; Posner & Keele, 1968; Reed, 1972; Rosch, 1973).

## 1.2 MATHEMATICAL CONCEPTS

Developing mathematical concepts is not unlike developing other concepts. Within mathematics education, both the classical and prototype views are often employed when addressing the formation of mathematical concepts. In line with the classical view, mathematical concepts generally have precise definitions ensuring mathematical coherence and providing the foundation for building mathematical theories. In mathematics, examples are absolute, determined by the canons of mathematical correctness. However, these same mathematical concepts may have been encountered by the individual in other forms prior to being formally defined. Even after they are defined, mathematical concepts often invoke images both at the personal as well as the collective level. Thus, for learners, some instances of a concept may be better examples than others. This is in line with the probabilistic view.

Within mathematics education, we may differentiate between a formal concept definition, a personal concept definition, and a concept image. A concept definition refers to "a form of words used to specify that concept" (Tall & Vinner, 1981, p. 152). A formal concept definition is a definition accepted by the mathematical community whereas a personal concept definition may be formed by the individual and change with time and circumstance. A personal concept definition may not obey the normative rules of mathematical definitions and may even be incorrect. The term concept image is used to describe "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). Because the concept image actually contains a conglomerate of ideas, some of these ideas may coincide with the definition while others may not. For example, a function may be formally defined as a correspondence between two sets which assigns to each element in the first set exactly one element in the second set. Yet, students may claim that a function is a rule of correspondence (Vinner, 1991). This image does not contradict the definition. However, it is limited and eliminates the possibility of an arbitrary correspondence. At other times, the concept image may include images which are inappropriate and contradict the concept definition. This is discussed in more detail when we focus later on geometry.

When a problem is posed to an individual, there are several cognitive paths that may be taken which take into consideration both the concept image and concept definition. At times, although the individual may have been presented with the definition, this particular path may be bypassed. Consider, for example, the question of whether zero is an even number, an odd number, or neither even nor odd. In one study, two sixth grade students claimed that zero was neither even nor odd (Levenson, Tsamir, & Tirosh, 2007). Both students knew the definition of even numbers as being divisible by two. Yet, one student's concept image of even numbers included being "built from twos" and she could not see how zero was built from twos. The second student's concept image of zero was that of it representing nothing and therefore could not be divided by two. Both students had a correct concept definition of even numbers. They had both previously claimed that 14 was an even number because it is divisible by two. In other words, they knew that even numbers are divisible by two. Yet, when considering zero, both students responded at first intuitively, according to their concept images, and not according to the acceptable concept definition. According to Vinner (1991), an intuitive response is one where "everyday life thought habits take over and the respondent is unaware of the need to consult the formal definition" (p. 73). Intuitive knowledge is both self-evident and immediate and is often derived from experience (Fischbein, 1987). As such it does not always promote the logical and deductive reasoning necessary for developing formal mathematical concepts. "Sometimes, the intuitive background manipulates and hinders the formal interpretation" (Fischbein, 1993a, p. 14). Recently, Stavy and Babai (2010) explored how intuitive processing of irrelevant quantities interferes with formal/logical reasoning in geometry. In their study, they investigated how adults compared the areas and perimeters of shapes in two conditions: (1) congruent conditions – where the response is in line with the intuition as the area of one shape is larger than the second shape and so its perimeter is also larger than the second shape and (2) incongruent conditions – where the correct response runs counter to the intuition as the area of one shape is larger than the second shape but its perimeter is not (see Figures 1a and 1b).



*Figures 1a. Congruent condition*

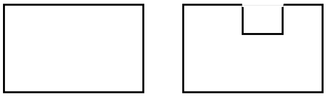


*Figure 1b. Incongruent condition*

Brain imaging suggested that executive control mechanisms might have a role in overcoming intuitive interference. They also point to the importance of noticing that although two tasks might be mathematically equivalent, they could, psychologically, be very different, i.e., the comparison of perimeter of an incongruent complex task is more demanding than the corresponding simple task.

The distinction in mathematics education research between intuitive thinking and behavior and analytical thinking and behavior may be complemented by considering general cognitive behaviors such as the dual-process theory of two parallel systems, System 1 (S1) and System 2 (S2) (Leron & Hazzan, 2006). S1 processes are "characterized as being fast, automatic, effortless, unconscious and inflexible…can be language-mediated and relate to events not in the here-and-now" (p. 108). S2 processes are "slow, conscious, effortful and relatively flexible" (p. 108). Consider the following mathematics problem presented to university students:

> A baseball bat and ball cost together one dollar and 10 cents. The bat costs one dollar more than the ball. How much does the ball cost? (Kahneman, 2002, p. 451)

Many students initially answered that the ball costs 10 cents. When students incorrectly answer a mathematics problem, it may not necessarily be due to lack of mathematical knowledge. When analyzing why students wrote an incorrect mathematics sentence for a given word problem, Leron and Hazaan (2006) concluded that it was not that the students lacked the necessary mathematics knowledge. Instead, the fault was most probably due to the general cognitive process whereby S1 took over too quickly for S2 to even have a chance. That is, S1 brought to mind the most easily accessible path which looked more or less correct while S2 failed in its role as a critic and monitor. They concluded by referring back to Fischbein (1987) in that students have to learn to be aware of the interactions between intuitions and the more formal meaning of mathematical concepts.

How are all the above theories related to young children? Although Tall and Vinner (1981) investigated their concept image-concept definition theory within the context of advanced mathematical thinking, the interplay between concept definition and concept image is part of the process of mathematical concept formation for young children as well. Young children learn about and develop concepts, including geometrical concepts, before they begin school. As such, their concept image is often limited to their immediate surroundings and experiences and is based on perceptual similarities of examples, also known as characteristic features (in line with Smith, Shoben, & Rips, 1974). This initial discrimination may lead to only partial concept acquisition in that children may consider some nonexamples to be examples and yet may consider some examples to be nonexamples of the concept. Later on, examples serve as a basis for both perceptible and nonperceptible attributes, ultimately leading to a concept based on its defining features. When a child has developed the mechanism which will allow the correct identification of all examples of a concept, as well as the exclusion of all nonexamples, we may conclude that the child has acquired that concept.

The interplay between the concept image and concept definition plays a major role in geometric concept formation (Vinner & Hershkowitz, 1980). In the next section we elaborate on this as we consider concept formation in geometry as well as theories related to the development of geometrical reasoning.

## 1.3 GEOMETRICAL CONCEPT FORMATION AND REASONING

Before considering concept formation in geometry, let us consider the nature of geometrical concepts. Fischbein (1993b) called the geometrical figures, figural concepts. In this he wished to convey their dual nature as both figures and concepts. Consider the following proof for why the base angles in isosceles triangle ABC are equal (see Figure 2a).

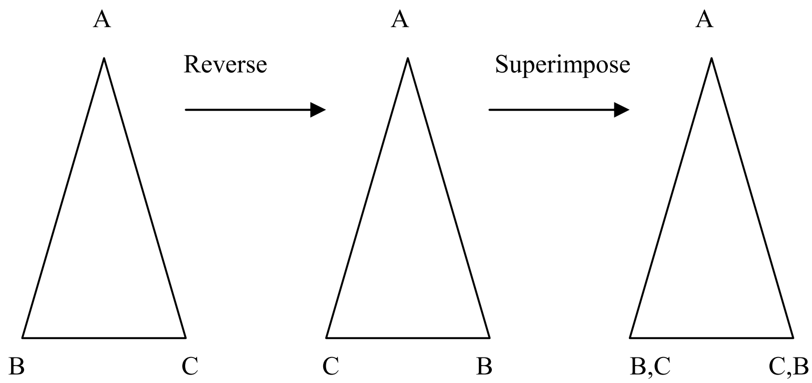



*Figure 2a.* *Figure 2b.* *Figure 2c.*

Imagine that you detach triangle ABC from itself, reverse it such that AC is on the left side and AB is on the right side (see Figure 2b), and superimpose it back onto the original one (see Figure 2c). Angle A remains the same; the lengths of AB and AC are equal so that the two sides coincide perfectly. The reversed triangle sits perfectly on the original triangle. Thus, we may conclude that the two triangle are congruent and therefore their corresponding angles are equal, leading to the conclusion that the base angles of an isosceles triangle must be equal. This proof takes into consideration both concepts and figures. Lines, points, and angles are ideal concepts. It is the image, which is manipulated. Yet, in reality, can we actually detach an object from itself? The objects we refer to are concepts. They are ideals. However, their intrinsic nature as figures allows us to consider their manipulation. Geometrical figures are concepts, abstract ideas derived from formal definitions. As such, geometrical entities do not actually exist in reality. As figures they have visual images. Images may be manipulated. To summarize, Fischbein (1993b) claimed that the figural concepts "reflect spatial properties (shape, position, magnitude), and at the same time, possess conceptual qualities – like ideality, abstractness, generality, perfection" (p. 143).

When conceptualizing children's formation of geometrical concepts, Piaget (e.g. 1956; 1960) took a cognitive developmental stand. That is, geometrical thought develops in stages following an experiential order which does not necessarily reflect the historical development of geometry. At the first stage, a child uses sensory-motor activities to explore space, constructing representations of topological concepts such as interior and exterior, without size or shape. At the second stage, the child develops concepts of projective geometry such as a straight line or a right angle. At the third and last stage, children discriminate location in two- and three-dimensional space succeeding with measurement and higher level tasks (Piaget, Inhelder, & Szeminska, 1960). At this stage, the child is ready to study notions of Euclidean geometry such as angularity and parallelism. In general, Piaget differentiated between topological and Euclidean figures and conceived of geometry as the study of space.

An extension of this view of the child's development of geometric concepts was put forth by van Hiele (1958). According to this view, with the support of instruction, students' geometrical thinking progresses through a hierarchy of five levels, eventually leading up to formal deductive reasoning. Consider the rectangle below in Figure X and possible responses to the questions: What type of figure is this? How do you know?

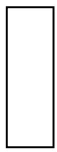


*Figure 3. Rectangle in vertical position.*

Child A: It is a rectangle because it looks like one.
Child B: It is a rectangle because it has four sides, two long sides and two short sides, and the opposite sides are parallel.
Child C: It is a rectangle because it is a parallelogram with right angles.
Child D: I can prove it is a rectangle if I know the figure is a parallelogram and has one right angle.

The first child, according to the van Hiele theory, is at the most basic level where students use visual reasoning, taking in the whole shape without considering that the shape is made up of separate components. Students at this level can name shapes and distinguish between similar looking shapes. The second student is at the second level where students begin to notice that different shapes have different attributes but the attributes are not perceived as being related. The third child notices the relationship between parallelograms and rectangles. This child is at the third level where relationships between attributes are perceived. At this level, definitions are meaningful but proofs are as yet not understood. The fourth child has reached a level of formal deduction, where students may establish theorems within an axiomatic system. The fifth level is rigor and formality. Some investigators have suggested a pre-recognition level, Level-0, at which level students may perceive shapes but only attend to a subset of a shape's characteristics (Clements, Swaminathan, Hannibal, & Sarama, 1999). For example, learners may be able to separate triangles from quadrilaterals, noting the difference between the number of sides the polygons have, but not be able to distinguish between different quadrilaterals. At this level, when asked to sort, for example, rectangles from non-rectangles, a student may not be able to correctly sort all the figures and will generally claim that some "look like doors" and other not.

As this book is concerned with young children's acquisition of geometrical concepts, we are mainly concerned with the first three van Hiele levels, as students move from visual reasoning to recognizing attributes and the relationships between attributes. In the following sections we elaborate on these stages including different factors which may impact on the acquisition of geometrical concepts.

*Level one: Visual reasoning and naming*

Visual reasoning begins with nonverbal thinking (van Hiele, 1999). Children judge figures by their appearances without the words necessary for describing what they see. For example, one study found that 5-year old children often identify as triangles, triangle-like shapes with curved sides, either convex or concave, similar to those shown in Figure 4 (Clements, Swaminathan, Hannibal, & Sarama, 1999).

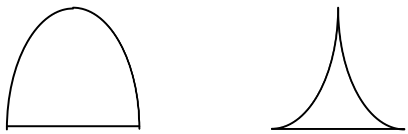


*Figure 4. Triangle-like figures with convex and concave sides.*

When reviewing the children's descriptions of circles, triangles, and rectangles, only a few children referred to the attributes of these shapes, indicating that most children were operating at the first van Hiele level of geometrical thinking.

Concept formation may also be linked to naming. For infants and very young children, the act of naming may serve as a catalyst to form categories (Waxman, 1999). In fact, categorization improves greatly when children hear a single consistent name for various examples of a category as opposed to hearing different names for the different examples (Waxman & Braun, 2005). Interestingly, Markman (1989) proposed that when children hear a new name for an object, they assume it refers to a whole object and not to its parts. This coincides with the first van Hiele level in which children first take the whole shape into consideration without regarding its components.

Studies have also shown that children assume a given object will have one and only one name (e.g. Markman & Wachtel, 1988). Thus, children operating at this level may reason that a square is not a triangle merely because it is a square and if they know the name of this shape to be a square then it cannot be a triangle (Tsamir, Tirosh, & Levenson, 2008). For example, when asked if the figure below (see Figure 5) is a triangle, Donna, a five-year old child answered, "No. It's an ellipse."

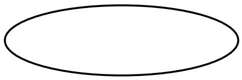


*Figure 5. Is this a triangle?*

For this child, it was enough to know that the figure is an ellipse to exclude the possibility of it being a triangle. While in this case, the child's reasoning led to a correct identification, it may also lead to confusion. Believing that all objects have one and only one name may contribute to the difficulties children have in accepting the hierarchal structure of geometric figures. When asked if the square in Figure 6 was a rectangle, Benjamin responded, "No, it is not a rectangle. It is a square." For this child, if the figure already has one name, a square, then how can it also be called something else?

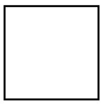


*Figure 6. Is this a rectangle?*

Visual reasoning was also discussed by Satlow and NewCombe (1998) who investigated children's identification of four shapes: circles, triangles, rectangles, and pentagons. For each shape they presented children with examples and nonexamples, which they termed valid and invalid instances. Valid instances were further categorized into typical and atypical instances. For example, the regular pentagon with horizontal base was considered a typical pentagon. A tall narrow pentagon was considered atypical. An open pentagon-like figure was invalid. Results indicated that children ages 3-5 rejected more of the atypical figures than the invalid figures. However, by the second grade a shift occurred whereby more of the children correctly rejected the invalid figures than the atypical figures.

Focusing on the specific shapes, children ages 4-5 correctly identified more atypical rectangles than atypical triangles and pentagons. Satlow and Newcombe suggested that the difference between the shapes may lie in their visual characteristics. The rectangle has limited variability of characteristic features. In contrast, triangles and pentagons may vary in the degree in their angles providing a wider variety of shapes. Symmetry and angle degrees may be considered attributes of figures. Some attributes, namely critical attributes, stem from the concept definition while others, non-critical attributes, do not. In the next section we discuss the difference between critical and non-critical attributes and their relationship to geometric reasoning.

*Levels 2 and 3: Critical and non-critical attributes*

At the second level, children discern between the attributes of figures. Attributes may be critical or not-critical (Hershkowitz, 1989). In mathematics, critical attributes stem from the concept definition. Definitions are apt to contain only necessary and sufficient conditions required to identify an example of the concept. Other critical attributes may be reasoned out from the definition. Hence, if we define a quadrilateral as a "four sided polygon", we may then reason that the quadrilateral is a closed figure that also has four vertices and four angles. The critical attributes then include (a) planar figure, (b) closed figure, (c) four sides, (d) four vertices, (e) four angles. Non-critical attributes include, for instance, the overall size of the figure (large or small), color, and orientation (horizontal base). Individuals who base their reasoning on critical attributes may at the very least be operating at the second van Hiele level. If the student points out that a figure is a quadrilateral because it is a polygon that has four sides and therefore it also has four angles and four vertices, then that child may be operating at the third van Hiele level. Recall that children operating at the third van Hiele level find definitions meaningful and perceive the relationships between attributes. Hershkowitz and Vinner (1983) and Hershkowitz (1989) also found that reasoning based on critical attributes increases with age.

While all examples of a concept must contain the entire set of critical attributes for that concept, sometimes children pay more attention to the non-critical attributes of different examples. For example, would the following figure be considered a square?

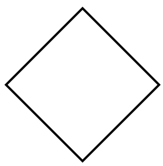


*Figure 7. Square rotated 45°.*

Burger and Shaughnessy (1986) found that although the orientation of a figure is a non-critical or irrelevant attribute, 3rd and 5th grade students may exclude the above figure from being a square because of its rotation.

Which of the following would be considered rectangles by children ages 4-6 years old?

*Figure 8. Which is a rectangle?*

Clements, Swaminathan, Hannibal, and Sarama (1999) found some children claimed that all three figures are rectangles because they are “long and skinny”. It seemed that if the long and skinny quadrilateral had at least one pair of parallel sides, children would accept the figure as a rectangle, paying less attention to perpendicularity.

A critical attribute of one figure may be a non-critical attribute of another. For instance, the critical attribute of equal measure when considering the four equal sides and four equal angles of the square, is a non-critical attribute when considering examples of a quadrilateral. In a follow-up study to Clements et al. (1999), Hannibal and Clements (2000) identified additional non-critical attributes. These included skewness and aspect ratio. For example, triangles, such as the one in Figure 9, that lacked symmetry or where the height was not equal to the width were not always identified as triangles. Rectangles, such as the one in Figure 9, that were too narrow or not narrow enough were also not accepted.

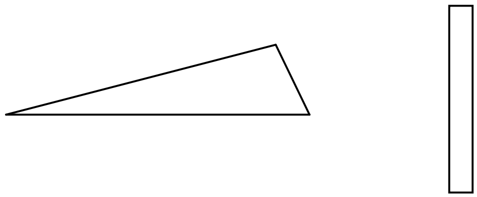


*Figure 9. “Skewed” triangle and a “too narrow” rectangle.*

*Prototypes and concept formation*

Recall that the probabilistic view of concept formation, discussed in the first section of this chapter, takes into consideration that some features are more characteristic or probable than others and thus some examples are 'better' examples than others. Ideal examples are called prototypes. Prototypical examples may play an important role in children's conceptual development. Initially, children's concept images consist of mostly prototypical examples. In drawing tasks, children most often draw a prototypical example. Hershkowitz (1989) found that even when an invented concept is introduced solely by a verbal definition, a prototypical shape emerges from students' drawings. In her study, students age 11-14 as well as both prospective and practicing elementary school teachers were given the following definition: A "bitran" is a geometric shape consisting of two triangles having a common vertex. They were then asked to draw two examples of this concept. Take a moment to draw an example of a "bitran". Results indicated that over 40% of the students and approxamitely 50% of the teachers drew the figure shown in Figure 10. In other words, the verbal definition elicited very similar concept images among all participants.

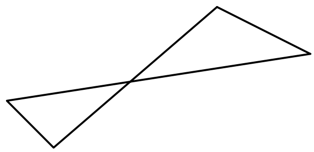


*Figure 10. Draw an example of a "bitran".*

Clements, Swaminathan, Hannibal, and Sarama (1999) suggest that different shapes may have different numbers of prototypes. They reported that the circle and square have fewer prototypes than rectangles and triangles. The data also suggested that children have a prototype of a rectangle which is long, for the most part disregarding orientation. Thus, many young children incorrectly identified a long parallelogram as a rectangle.

Regarding reasoning about shapes, when analyzing the children's verbal responses to identification tasks of various geometric figures and their descriptions of those figures, it was found that many children compared the shapes to visual prototypes. Using the prototypical triangle as a reasoning tool was demonstrated by Martin, Lukong, and Reaves (2007). They found that when kindergarten children were given a paper with several drawn figures, various triangles in different orientations, along with various non-triangles, and given the task of identifying all the triangles on the paper, children were more likely to rotate the paper when identifying non-prototypical triangles than when identifying prototypical triangles. In addition, when asked to make non-triangles into triangles, more children were likely to draw a prototypical triangle on top of the shape given than just "fix" the missing or incorrect attribute. For example, when children were told to "fix" a triangle-like shape with concave sides, they tended to draw on top it a prototypical triangle (see Figure 11).

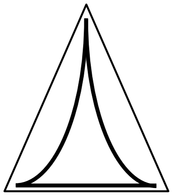


*Figure 11. Prototypical triangle superimposed onto triangle-like figure with concave sides.*

Some studies have suggested that over exposure to prototypes may impede the growth of fuller concept acquisition. For example, Kellogg (1980) suggested that prototypes are formed when certain non-critical attributes of a shape appear frequently in examples and students begin to associate these non-critical attributes with examples of the shape. Thus, if children mostly see equilateral triangles in an ‘upright’ position, they may mistakenly believe that having equal length sides is a critical attribute of all triangles and that being in an ‘upright’ position is also a critical attribute. In such a case, the child may not accept a right triangle or a scalene triangle as examples of triangles. Wilson (1986) advocated the use of nonexamples in order to lessen the effect of prototypes. For example, if a child is presented with many non-triangle figures that have equal sides he may come to realize that having equal sides is not a critical attribute of a triangle (see Figure 12). By exposing students to nonexamples with the same non-critical attributes, students may begin to differentiate between critical and non-critical attributes.

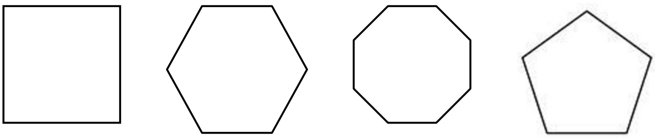


*Figure 12. Nonexamples of triangles that have equal sides and equal angles.*

It is often the non-critical attributes which contribute to the makings of a prototypical example. Hershkowitz (1989) claimed that in addition to the necessary and sufficient (critical) attributes that all examples share, prototypical examples of a shape have special (non-critical) attributes “which are dominant and draw our attention” (p. 73). The prototypical examples often have the longest list of attributes. Consider for example, the square. Its critical attributes include: closed polygon, four sides, four vertices, four angles, opposite sides that are parallel, sides that are all equal measure, angles that all measure 90°, diagonals which bisect each other, diagonal which are equal measure. A subset of these critical attributes, namely closed polygon, four sides, four vertices, and four angles, is the set of critical attributes for quadrilaterals. Thus the hierarchy of quadrilaterals is reversed when considering their critical attributes (see Figures 13a and 13b). While the set of quadrilaterals includes squares, the set of critical attributes of the square includes the set of critical attributes of quadrilaterals.

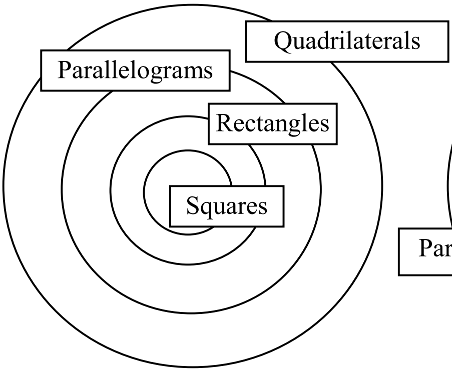



*Figure 13a. Hierarchy of quadrilaterals.*

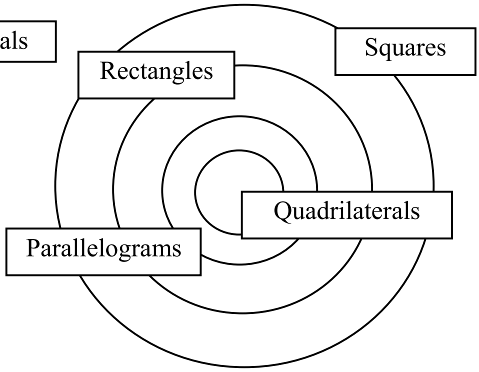



*Figure 13b. Hierarchy of quadrilateral attributes.*

Smith, Shoben, and Rips (1974) argued that prototypical examples are rapidly identifiable as an example of the category, whereas other examples may take longer to identify. They also hinted at the possibility that some nonexamples are so dissimilar that they are rapidly identified as being nonexamples of the category. Could there be prototypical nonexamples? This question was raised by Tsamir, Tirosh, and Levenson (2008) when they found that some figures were rapidly and without question identified as nonexamples for triangles. In other words, they were intuitively recognized as nonexamples. The interplay between intuition and geometric thinking is discussed further in the next section.

*Intuition and geometrical concept formation*

In the second section we discussed the possible conflict between the concept image and concept definition (Vinner, 1991). Similarly, Fischbein (1993a) described the possible conflicts, contradictions, and internal tensions which may arise as children grapple with both the intuitive and formal nature of figural concepts. “An intuitive cognition is a kind of cognition that is accepted directly without the feeling that any kind of justification is required. An intuitive cognition is then characterized, first of all, by (apparent) self-evidence” (p. 232). The formal nature of mathematics refers to axioms, definitions, theorems, and proofs. These need to be actively used by the student when reasoning about and within mathematics.

Consider, for example, the following figures (see Figure 14):

*Figure 14a.*

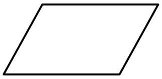


*Figure 14b.*

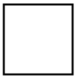


*Figure 14c.*

*Figure 14d.*

Which of these figures are parallelograms? Which of these figures would a child consider to be a parallelogram? Which of these figures would the child automatically identify as a parallelogram and which would need explaining?

Although a child may be aware of the definitions for various quadrilaterals, the figure may promote an intuitive response, one which is immediate and where the child feels little need to justify himself. This may be the case when identifying Figure 14b as a parallelogram. At times, the coercive nature of intuitive cognitions is such that the figural particularities may be so strong as to annihilate the effect of the formal constraints. Thus, a child may claim that the long trapezoid in Figure 14d is a parallelogram noticing the pair of parallel sides and ignoring that the definition calls for two pairs of parallel sides. It also might be the case that definitions are ignored when the figure has extra non-critical attributes. This is might be the case with the rectangle in Figure 14a and the square in Figure 14c. Even though a child may know the definition of a parallelogram, he may not accept that a rectangle and a square are parallelograms. At times, intuitive cognitions fall in accordance with mathematical truths, as with Figure 14b, and at times, they contradict these truths, as wtih Figure 14c. Fischbein concluded that a major task of mathematics educators is to help students cope with the interaction between the formal and intuitive constraints of the figural concepts and that instruction could and should shape and form mental processes.

*Are the van Hiele levels discrete?*

As the van Hiele levels extended Piaget's theory, it was originally thought that these levels were discrete. Recently, however, research has suggested that the van Hiele levels may not be discrete and that a child may display different levels of thinking for different contexts or different tasks. For example, Burger and Shaughnessy (1986) claimed that reference to non-critical attributes often points to an element of visual reasoning. Thus, a child using this reasoning may either be operating at van Hiele level one or at van Hiele level two or perhaps at both levels concurrently. If the child is employing visual reasoning, we would say that he is operating at the first level. On the other hand, pointing to a specific attribute, albeit a non-critical attribute, and not judging the figure as a whole, may point to reasoning at the second level. Comparing a figure to the prototypical examples is what Hershkowitz (1990) called prototypical judgment. This may be partly a visual judgment as the "prototype's irrelevant attributes usually have strong visual characteristics" (p. 83). Clements and Battista (2001) suggested that the van Hiele levels of geometric reasoning may even develop simultaneously, albeit not necessarily at the same rate. Taking all of this into account we suggest that reasoning based on non-critical attributes may serve as a bridge between the first and second van Hiele levels of thought. In general, the level at which a child operates may be influenced by his age, experience, and the nature of the task. Whether or not the van Hiele levels are discrete or not, whether or not a child may operate on two levels at the same time or not, it is helpful to characterize children's geometric thinking according to these levels. The van Hiele model allows us to assess children's geometric reasoning and plan lessons that will guide students towards using only critical attributes as the deciding factor in identifying examples. In turn we move towards one of our major goals in mathematics education, that of developing concept images that are in line with the concept definitions.

In this section, we discussed the development of geometrical concepts focusing on two-dimensional figures. In the next section, we discuss research related to three-dimensional figures.

*Developing three-dimensional concepts*

Much of what has been previously discussed regarding two-dimensional figures may be applied to three-dimensional figures. For example, research has related to the possibility of extending the van Hiele levels to three-dimensional shapes (Gutiérrez, Jaime, & Fortuny, 1991). As such, at the first level, solids are judged based on the whole without consideration to the components. At the second level, children identify attributes such as the number of faces and the shape of faces, but do not perceive the relationship between attributes. At the third level, definitions are meaningful and students can logically classify solids based on the relationship between attributes. At the fourth level, students are able to prove theorems related to three-dimensional geometry. Regarding reasoning about three-dimensional shapes, Aubrey (1993), noted that children explore and build with three-dimensional objects and describe regular three-dimensional shapes with the same mixture of formal and informal responses that are given for two-dimensional shapes.

Other studies pointed to the use of plane geometry terminology when young children describe three-dimensional figures. For example, one study found that first graders often refer to a cube as a "square" (Nieuwoudt & van Niekerk, 1997). Other children described solids as "pointy" or "slender", using terminology more appropriate for two-dimensional figures (Lehrer, Jenkens, & Osana, 1998). On the other hand, three-dimensional objects are tangible and thus may elicit additional modes of concept formation. For example, Roth and Thom (2009) described an episode where second graders were learning about three-dimensional objects by manipulating them and thus experiencing the objects in different ways. For example, one child picked up a cylinder, looked at it from different perspectives, put it down on the table and picked it up again. It was also compared to other, different size cylinders. The child experienced and described the cylinder as an object which is round, may be rolled between the palms of hands, has circular flat ends, and feels smooth. According to their theory, the general concept of a cylinder was formed from the multitude of experiences which could then be activated by any one experience. Gutiérrez (1996) claimed that handling real three-dimensional solids may not be enough to acquire these concepts because rotations made with hands are usually done rather quickly and unconsciously, so that children, especially young children, may hardly be able to reflect on the actions.

## 1.4 LOOKING AHEAD

In this chapter we discussed theories and research related to the development of concepts, mathematical concepts, and geometrical concepts among children. These theories form the background for the following chapters. The next chapter focuses on the development of the concept of a triangle. We use triangles as a basic figure to illustrate how children may come to develop a concept image of a polygon that correlates with the concept definition of that polygon. In other words, as you read about triangles, you may imagine how the same may be said for pentagons or hexagons.

CHAPTER 2

# WHAT DOES IT MEAN FOR PRESCHOOL CHILDREN TO KNOW THAT A SHAPE IS A TRIANGLE? BUILDING CONCEPT IMAGES IN LINE WITH CONCEPT DEFINITIONS

Consider the following scalene triangle:

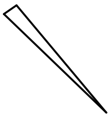


*Figure. 1. Scalene triangle.*

Dan (age 3), Nancy (age 4), and Jordan (age 5) learn in different preschools. Each child was presented with a drawing of the same scalene triangle shown in Figure 1 and was requested to answer the following two questions: (1) Is this a triangle? (2) Why? They responded:

Dan: Yes, because it has vertices.[1]

Nancy: Yes, because it has three vertices and three straight lines.

Jordan: Yes, because it has three vertices, three sides, and it's closed.

Each child was also presented with the following non-triangle shape (see Figure 2) and again asked: Is this a triangle? Why?

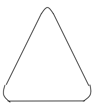


*Figure 2. Rounded non-triangle shape.*

They responded:

Dan: No, it doesn't have vertices.

[1] The word "vertices" is a literal translation from the Hebrew, "kodkod", not to be confused with corners or points.

> Nancy: No. It doesn't have vertices like this one (points to the previous triangle). It's like a triangle. It has three sides but no vertices.
>
> Jordan: No. It doesn't have vertices. It only has rounded corners.

Are you surprised by the children's judgments? Are you surprised by their justifications? As discussed in the previous chapter, young children mostly operate at the first van Hile level, relying on visual reasoning, taking in the whole shape when identifying examples and nonexamples of geometrical shapes. One would think that when confronted with the shapes in Figures 1 and 2, young children would not so readily identify correctly the scalene triangle in Figure 1 and that the rounded non-triangle would be incorrectly identified as a triangle. Yet, the children's responses above indicate that it is possible for children, even as young as three years old, to incorporate critical attributes when identifying examples and nonexample of triangles. Although the above children learned in different preschools, all three preschools participated in enrichment programs that included professional development for the teachers as well as extra enrichment for the children within the preschool itself.[2]

In this section we discuss how young children may develop a concept image of triangles in line with the concept definition of triangles. We focus on two key elements – identifying examples and nonexamples of triangles and explaining why an example is, or a nonexample is not, a triangle.

## 2.1 IDENTIFYING TRIANGLES – ARE ALL EXAMPLES AND NONEXAMPLES CREATED EQUAL?

In their study of two kindergartens and the triangle activities presented in each kindergarten, Tirosh and Tsamir (2008) described how two kindergarten teachers from the same community, Yardena and Anat (pseudonyms), wanted to investigate if their 4-5 year old children could identify triangles. Each teacher drew up a set of figures, one figure to a card, and asked each child if the figure was or was not a triangle. To their surprise, Yardena found that the children in her class were quite capable of identifying triangles but the children in Anat's class were not. How could this be? In Figure 3 we present the shapes each teacher showed her children (as they appear in Tirosh & Tsamir, 2008, p. 11).

Taking a close look at the examples of triangles each teacher presented in her class, we note that Yardena only presented to her children equilateral or isosceles triangles with a horizontal base and "right side up". That is, she presented to her children prototypical examples, intuitively accepted as such by the children. On the other hand, Anat presented to her children one equilateral triangle with a horizontal base, one "upside down" isosceles triangle, a right triangle, a scalene triangle, and an obtuse triangle. No wonder the teachers' investigations led to such different results.

[2] The preschool for 3 year old children participated in the program, *First Steps in Mathematics*, run in collaboration with WIZO. The preschools for 4 and 5 year old children participated in the program, *Starting Right: Mathematics in Kindergarten*, initiated in collaboration with the Rashi Foundation.

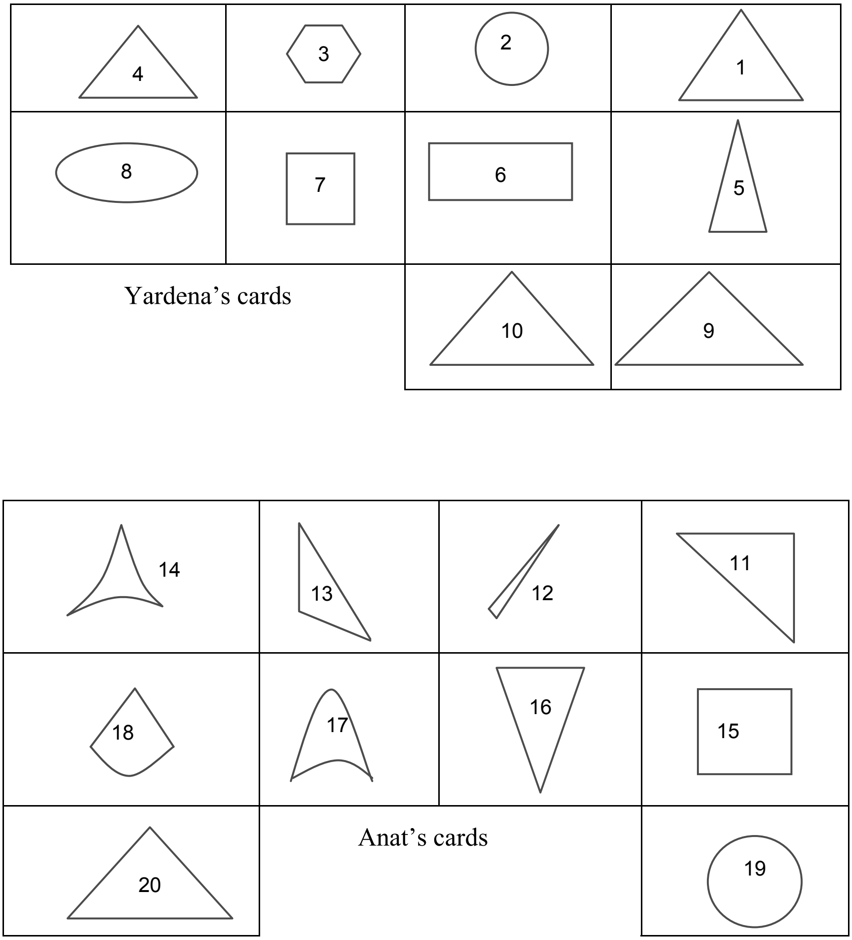



*Figure 3. Yardena's and Anat's cards.*

Now take a closer look at the nonexamples presented by each teacher. Yardena's nonexamples consisted of mostly familiar shapes like a circle and square; and even if one claims that the hexagon was not familiar to children, it certainly does not resemble the overall shape of a triangle. Anat's nonexamples consisted also of a circle and a square. However, she included other shapes that were visually similar to triangles, in a holistic way. In other words, Yardena's nonexamples were visually far removed from triangles while Anat included some shapes that visually resembled triangles.

The children in Anat's class only identified the prototypical equilateral triangle as a triangle. They did not identify the other triangles as triangles. Most children incorrectly identified shapes 14, 17, and 18 as triangles. Reverting to Tall and

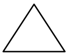
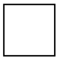
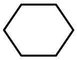
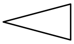
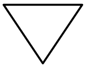
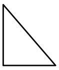
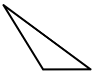
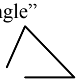
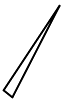
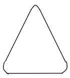
Vinners' (1981) concept image-concept definition theory discussed in the previous chapter, we may infer that the children in Anat's class had a concept image limited to the prototypical triangle. Regarding the children in Yardena's class, we cannot know what their concept image is as these children were only presented with prototypical triangles and with nonexamples that were visually far removed from triangles.

As the above study illustrates, an important element of what it means to know triangles is being able to identify a wide variety of examples and nonexamples. We have also illustrated that not all examples and nonexamples are created equal. That is, although all examples share the same necessary and sufficient critical attributes, a prototypical example has special (non-critical) attributes "which are dominant and draw our attention" (Hershkowitz, 1989, p. 73). Mathematically, all examples are equal. However, psychologically, they may not be identified with equal ease. Prototypical examples often have the longest list of attributes. Smith, Shoben and Rips (1974) argued that some examples are rapidly identifiable as an example of the category, whereas other examples may take longer to identify. In other words, some examples are intuitively accepted as representative of the concept in that they are accepted immediately, with confidence, and without the feeling that any kind of justification is required (Fischbein, 1987). Regarding the identification of non-triangles, it was found that first and third grade students identified intuitive nonexamples of triangles in a shorter time than it took them to identify non-intuitive nonexamples of triangles (Spector, 2010).

Identifying which examples and nonexamples may be intuitively recognized as such is an important first step in building appropriate concept images. In our study of 42 children ages 4-5 years old (Tirosh, Tsamir, & Levenson, 2010), and 65 children ages 5-6 years old (Tsamir, Tirosh, & Levenson, 2008), different geometrical figures, each figure printed on a separate card, were presented one at time to children. Each child was asked if the figure was a triangle. These children learned in preschools where the teachers had not attended professional development courses in mathematics and where no special or extra mathematics enrichment was provided. Among the figures were seven examples and seven nonexamples of triangles (see Table 1). Examples were chosen to include prototypical as well as non-prototypical triangles. Following Hershkowitz (1990) the equilateral and isosceles triangles were considered to be prototypical examples. The other five examples were not considered prototypical. For example, Burger and Shaughnessy (1986) found that young children did not identify as a triangle a long and narrow triangle, such as the scalene triangle even when they admitted that the figure had three points and lines.

Results showed that indeed the equilateral and isosceles triangles presented "right side up" and with a horizontal base, were identified correctly and immediately by the vast majority of children. It is interesting to note that regarding the examples, the declining order of frequencies was the same for both age groups of children and that the isosceles and equilateral triangles with a different orientation were identified correctly in more instances than triangles with varied size angles and sides. This suggests that orientation may be less problematic than

*Table 1. Frequency (in percents) of immediate correct identification.*

| Triangles | 4-5 year olds (N= 42) | 5-6 year olds (N=65) | Non-triangles | 4-5 year olds (N=42) | 5-6 year olds (N=65) |
|---|---|---|---|---|---|
| Equilateral  | 88 | 98 | Square  | 97 | 100 |
| Isosceles  | 83 | 94 | Hexagon  | 100 | 100 |
| Sideways  | 62 | 51 | Ellipse  | 100 | 100 |
| Upside down  | 60 | 48 | Pentagon  | 88 | 82 |
| Right  | 48 | 42 | Zig-zag “triangle”  | 80 | 82 |
| Obtuse  | 19 | 20 | Open “triangle”  | 71 | 80 |
| Scalene  | 5 | 11 | Rounded “triangle”  | 7 | 5 |

the size of the angles or sides. It also makes sense. When a triangle is presented in a non-prototypical orientation, many children will rotate the triangle, orienting it to fit the prototypical image (Martin, Lukong, & Reaves, 2007). Thus, the orientation may be changed. On the other hand, the size of angles and sides may not be changed. We also note, however, that if we focus on the sideways and upside down triangles, it seems that the older group was more reluctant than the younger group to accept triangles with a different orientation. Possibly, the more experience children have with prototypical shapes and orientation, the more reluctant they are to accept differences. Burger and Shaughnessy (1986) noted that even among high school students, orientation could be an obstacle for correct identification.

We now consider the nonexamples. The nonexamples were all two-dimensional shapes gathered from three categories: commonly recognized geometrical shapes (other than triangles), uncommon geometrical shapes (other than triangles), and “almost triangles”. In the first category was the square, regular hexagon, and ellipse. Many current national curricula around the world explicitly state that preschool children (ages 3-6) should recognize and use the mathematical names for shapes. For example, in the U.S., the Curriculum Focal Points (NCTM, 2006) state that kindergarten children should identify by name “a variety of shapes such as squares, triangles, circles, rectangles, (regular) hexagons, and (isosceles) trapezoids presented in a variety of ways” (p. 12). In Israel, the National Mathematics Preschool Curriculum (INMPC, 2008) recommends that children ages 4-6 years identify by name triangles, circles, quadrilaterals, pentagons, and hexagons. At a later stage they recommend adding non-common figures such as ellipses and semi-circles. In the second category, uncommon geometrical shapes, is the pentagon. The pentagon used in this study is non-prototypical of pentagons. It was positioned with a horizontal base, in a similar manner as the prototypical triangle, and was elongated in such a manner as to visually suggest a triangle. The third category, “almost triangles” consisted of shapes that have one or more attributes missing but otherwise share most of the attributes of the prototypical triangle. In this category are the open “triangle”, rounded “triangle”, and the zig-zag “triangle”. The open “triangle” is missing the critical attribute of being a closed figure. The rounded “triangle” is missing vertices. The zig-zag “triangle” has jagged sides. On the other hand, all have horizontal bases and all have the illusion of threeness. Some of these figures have been investigated in other studies. For example, Hasegawa (1997) found that the rounded “triangle” is often identified as a triangle. Regarding the open “triangle”, some studies have shown that “openness” is regarded by many students to disqualify a figure from being a polygon (Hershkowitz & Vinner, 1983) while others have found that it is not necessarily a disqualifier (Rosch & Mervis, 1975). The zig-zag “triangle” was a figure created for this study. It is a 15-sided polygon. Yet is has the illusion of a triangle with jagged sides. Taken all together, the group of non-triangles afforded us the opportunity to investigate what makes a non-triangle intuitively accepted as such.

Referring back to Table 1, we first note that more children correctly identified the nonexamples than the examples. Among the nonexamples, the square, hexagon, and ellipse were immediately identified as nontriangles by all of the children in both age groups, except for one. This was not surprising. After all, we had taken into consideration that all preschool teachers following the national curriculum would present children with these shapes. In fact, a little more than half of the children responded to the square by simply identifying this figure correctly as a square, which apparently was enough to exclude it from the category of triangles. As mentioned in the first chapter, the act of naming may be considered a form of categorizing (Waxman, 1999). In addition, if we visually consider the whole shape, these three figures, as opposed to the other four nonexamples, are very dissimilar to the prototypical triangle. On average, approximately 80% of the children correctly identified the non-prototypical pentagon, the zig-zag "triangle", and open "triangle" as non-triangles. Finally, an average of 6% of the children identified the rounded "triangle" as a non-triangle. This is consistent with Hasegawa's (1997) findings, as mentioned above.

To summarize this section, we note that not all examples are recognized as such by preschool children and indeed not all nonexamples are recognized as such by preschool children. Watson and Mason (2005) coined the term "personal example space" to describe the collection of examples that is accessible to a person at a given time in a given circumstance and the interactions between these examples. We believe that a personal "nonexample space" may also exist. Often, learners have a very limited collection of examples as well as nonexamples in mind. We suggest dividing the personal example and nonexample space of a figure along two dimensions: a mathematical dimension and a psycho-didactical dimension (see Figure 4). In the case of triangles, the mathematical dimension divides the figures into examples and nonexamples of triangles according to the concept definition. The psycho-didactical dimension divides the figures into what is and is not intuitively identified as triangles and non-triangles according to the child's current concept image. The results of the above study may then be organized as in Figure 4. We argue that a significant aim of learning mathematics is extending and enriching the space of examples and nonexamples to which one has access. In order to promote this extension, it is necessary to take into consideration children's reasoning. This is discussed in the next section.

## 2.2 REASONING ABOUT TRIANGLES

Promoting correct identification of intuitive and nonintuitive examples and nonexamples should go hand in hand with promoting geometrical reasoning. Correctly identifying triangles and nontriangles is one element of knowing triangles. Equally important is being able to explain why some figure is or is not a triangle. Let us revisit the three children quoted in the beginning of this chapter. All three children identified correctly the scalene triangle. Moreover, all three mentioned one or more critical attributes of a triangle. In other words, when identifying triangles, these children were capable of operating at the second level of van Hiele reasoning, breaking up the shape into attributes. Yet, it is not enough to notice the attributes of a geometrical shape. As mentioned in chapter one, attributes may be critical or non-critical and identifying a geometrical shape must be based solely on the critical attributes, derived from the definition.

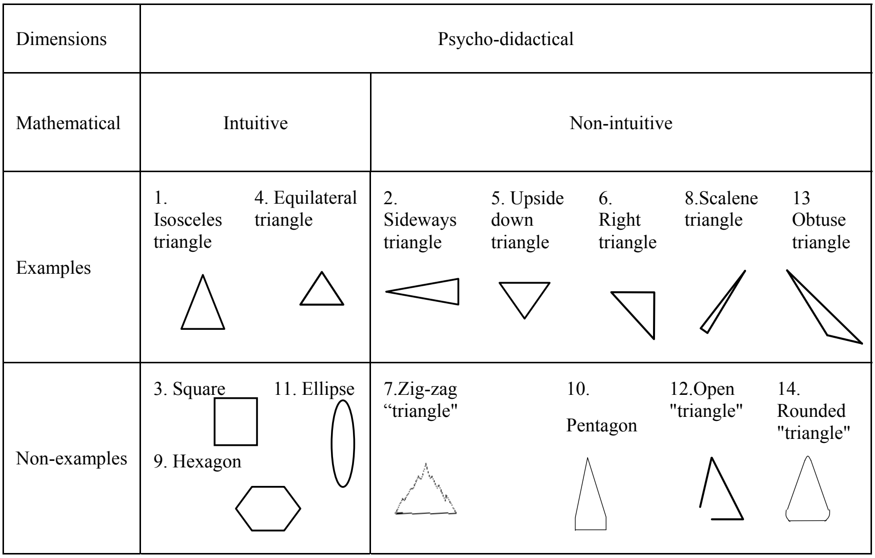



*Figure 4. Intuitive and non-intuitive examples and nonexamples of triangles.*

As mentioned in the first chapter, Fischbein (1993b) noted that the figural concepts comprise both intuitive and formal aspects. The image of the figure promotes an immediate intuitive response. Yet, geometrical concepts are abstract ideas derived from formal definitions. The interaction between the image and the abstract idea promotes both visual and attribute reasoning. Tsamir, Tirosh, and Levenson (2008) further differentiated between visual reasoning that takes into consideration the whole shape, visual reasoning that includes naming the figure, attribute reasoning that refers to critical attributes, and attribute reasoning that refers to non-critical attributes. Table 2 (Tsamir, Tirosh, & Levenson, 2008, p. 88) shows examples of each type of reasoning.

The categories in the table were then used to describe kindergarten children’s reasoning regarding nonexamples. They noted that most reasons were based on critical attributes, followed by, in decreasing preference, naming the figure, whole shape reasoning, and reasoning based on non-critical attributes. However, when comparing the combined results of the two categories representing visual reasoning with the combined results of the two categories representing attribute reasoning, more reasons were based on visual cues than on specific attributes.

*Table 2. Coding reasons after identifying a figure*

| Category | Reasons |
| --- | --- |
| Purely visual reference to the whole figure | "It looks (doesn't look) like a triangle."<br>"You see (don't see) the shape."<br>Traces the figure without saying a word. |
| Naming | "It's a rhombus (or some other geometric shape – correct or incorrect)."<br>"It's a bonfire (names an object)." |
| Reference to non-critical attributes | "Because this (points to a particular side) is too small (short, big, long)."<br>"It's (referring to the figure) too thin (fat, long, sharp)." |
| Reference to critical attributes | "It has three (four, five, many, none) sides (lines, points, corners)."<br>"It has to be closed."<br>"It has three rounded points." |

Focusing on the specific nonexamples provided some interesting insights regarding the relationship between reasoning and nonexamples. For example, reasoning regarding the square, ellipse, and hexagon was mostly based on ability to name the shape. When children did not know the correct name for one of these shapes, they provided imaginary names such as a mirror or an egg for the ellipse. Looking at the non-prototypical shape of the pentagon, an exception to the general trend was observed. Whereas for the other non-triangles, no more than 6% of the reasons were based on non-critical attributes, when it came to the pentagon, 28% of the responses consisted of this type of reasoning. Furthermore, this type of reasoning consistently went along with correct identification of this figure as a non-triangle. Recall that the pentagon was a non-prototypical pentagon and was actually constructed to be somewhat similar to a triangle. Typically, children who used this type of reasoning commented on the figure's thinness or stretched out look. Even when children used critical attribute reasoning for this shape, their reasoning was often incorrect. For example, one child who correctly identified the pentagon as a non-triangle claimed "the sides are crooked." In other words, this child knew that a triangle must have three straight sides and interpreted the two sides on the left and the two sides on the right as just one side on the left and one on the right.

In the group of "almost triangles", more responses (over 35%) consisted of visual reasoning based on the whole figure for these non-triangles than for any of the other non-triangles. This type of reasoning led to correct or incorrect identification depending on whether the child thought that it looked like a triangle, or not. The exception in the group was the zig-zag "triangle". This figure stimulated the children's imagination. More responses (33%) consisted of naming this figure as some object (a bonfire, mountain, or thorn bush) than was done for any of the other figures in this study. This kind of reasoning was usually accompanied by a correct identification.

An important result in the sub-group of “almost triangles” was that considerably more reasons were based on critical attributes when identifying these figures than for the other non-triangles. This result was especially notable for the open “triangle”, where 62% of the responses included this type of reasoning. Yet, this reasoning was not always accompanied by a correct identification. Some children simply stated that “it’s still a triangle, even if it’s open.” Interestingly, 20% of the reasons referred to the amount of vertices being less than three. This second comment actually shows that some children knew that a vertex must be the connection of two segments and not just the end point of one segment.

Regarding the rounded “triangle”, 42% of the critical attribute reasons focused on the three “sides” of the “triangle”. These were consistently associated with an incorrect identification. The rest focused on three “points” or “corners”. While most children did not comment on the roundness, four children pointed to the three rounded corners and claimed, “it has three corners even though it’s rounded.” These children did not regard roundness as disqualifying the figure from being a triangle.

When considering the way the group of “almost triangles” was constructed, the fact that more children based their reasoning on critical attributes for this group than for the other two groups is especially interesting. The zig-zag “triangle” was missing one, possibly two critical attributes, depending on the focus of the child. As illustrated in Figure 5, zooming in, the zig-zag “triangle” had more than three vertices and sides. Zooming out, the zig-zag “triangle” had two “sides” that were not straight.

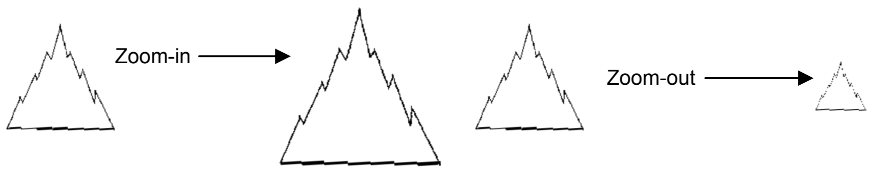



*Figure 5. “Zooming in” and “zooming out” the zig-zag “triangle.*

The rounded “triangle” was missing vertices. Yet, more children focused on the critical attribute of openness than on the other missing critical attributes. This raises two questions: Are all critical attributes equal in the eyes of children? Is it more noticeable when an attribute is missing than when it is there but in a deformed manner?

Reasoning with critical attributes is a necessary step in the child’s development of geometrical concepts. The study described above suggests that young children, even those who do not attend a preschool with an especially enriched geometrical environment, employ reasoning with attributes. Yet, as we also saw, this type of reasoning is not sufficient. A child may focus on the sides of a triangle but discount the rounded corners as not being important. How can we bring children to consider all of the critical attributes of a figure? How can we promote children to build concept images in line with concept definitions?

## 2.3 BUILDING CONCEPT IMAGES IN LINE WITH CONCEPT DEFINITIONS: THE POWER OF A WORKING DEFINITION

We believe that the key to bringing children's concept image of a figure closer to the concept definition for that figure is to promote the use of a definition as the decisive criterion for determining if an object is an example of a given concept. In geometry, specifically, we allow that visual judgment may be a necessary first level, but analytical judgment based on critical attributes should follow.

If the key to developing geometrical concepts in line with geometrical definitions is to promote the use of a definition, then of utmost importance is choosing a mathematically correct definition of a triangle appropriate for preschool children. What do we mean by appropriate? Consider the following definition of a triangle: A triangle is a three-sided polygon. It seems obvious that the word polygon may be problematic for young children. But the word polygon is problematic not only because it is unknown but because it infers within it other critical attributes. A polygon is a closed figure made up of sides. A triangle, like any polygon must be closed. It also must be made up of straight and not curved sides. The critical attributes of having straight sides and being closed are implicit in the term polygon, rather than being explicit. An additional problem with the above definition is that it makes no mention of vertices. Of course, mathematically, if a figure has straight sides and is closed then it follows that it necessarily has vertices. In addition, if a figure has three straight sides and is closed then it follows that it has three, and not four or five, vertices. However, this type of reasoning is more prevalent for older children operating at the third and fourth van Hiele levels and not the young children in preschool.

For preschool children, a minimal definition may be a disadvantage. Rather, one approach, that we chose to use in our work with young children, is to develop a working definition, a definition that children can use, that points to all the critical attributes, that children can refer to and check back with when examining a geometrical figure. Thus, although a triangle may be defined as a three-sided polygon, we use an expanded definition for young children which explicitly points to all the critical attributes of a triangle: a triangle is a closed figure which has three straight sides and three pointed vertices. This definition stresses that a triangle must be closed. It must have straight and not curved sides. It must have pointed and not rounded vertices. It must have three, and only three, sides and vertices. There is no mention of a polygon.

Another, equally important feature of our definition is its use of mathematical language. We do not substitute the word corner for vertex. Keeping in mind that knowledge built during preschool will follow the children throughout elementary school, we believe that it is important to build accurate foundations from the beginning. Although it may seem that the word corner is more child-friendly than vertex, a corner is not a well defined mathematical term. Corners may be rounded. A vertex may not.

By presenting children with this definition of a triangle we are presenting them with a reasoning tool. Of course, children must learn how to use the tool. They must also learn the meaning of each term in the definition and how to check each figure against the definition. This brings us back to the issue of examples and nonexamples. In the beginning of this section we pointed out the necessity of presenting children with both intuitive and non-intuitive examples and nonexamples. Here we add that the order and combination in which examples and nonexamples are presented may be used to illustrate to children the various critical and non-critical attributes of a triangle and encourage the use of a working definition as tool.

Let us begin with the critical attribute of pointed vertices. If we want the child to learn the meaning of a vertex and that it must be pointed rather than rounded, we may present to a child the following two figures (see Figure 6):

*6a: Prototypical triangle* *6b: Rounded corner "triangle"*

*Figure 6. Illustrating pointed vertices.*

The first is a prototypical triangle, intuitively recognized as such. The second is visually similar to the first. It is approximately the same size. It has the same orientation with horizontal base and is "right side up". They both have the quality of "threeness". The difference between the figures is that the figure on the left has vertices and the figure on the right does not. Young children may ignore this difference at first. As noted previously in Table 1, the triangle-like figure with rounded figures was the figure for which the least amount of children in both the pre-kindergarten and kindergarten group offered a correct identification. Consider the following statements regarding the second figure given by 5-6 year olds who had not attended preschools participating in mathematics enrichment programs. These children claimed that the second figure was a triangle and explained their identification:

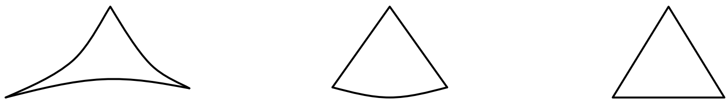
C1: It is a triangle because it looks like a triangle.
C2: It is a triangle because it has three sides.
C3: It is a triangle because it has three corners even though they're rounded.

Regarding C1, we cannot know if the child noticed the rounded corners or not. We do know that his explanation displays visual reasoning taking in the whole figure at once without relating to any attributes. The second child, relating to three sides, displays critical attribute reasoning. Yet, he makes no mention of the vertices. He has either not noticed the missing vertices or has noticed them and discounted them as not being critical. C3 has noticed the rounded corners but claims that roundness is not critical. Now consider C4:

C4: It is not a triangle because it has three sides but it doesn't have vertices.

C4 was a 5-year old learning in one of preschools participating in our program. His response indicated that he was aware of the working definition of a triangle and how to use the definition as a tool. On the one hand, he pointed out three "sides". Yet, despite that in his eyes the triangle had three sides, it was missing the vertices and therefore could not be a triangle. This is a significant step in the development of geometrical, and perhaps all mathematical concepts. This child was aware that if even one critical attribute was missing, then the figure or instance presented must not be an example of a triangle. In other words, although the figure may look like a triangle, it is missing the critical attribute of having pointed vertices, which is enough to discount it as being a triangle.

In addition to paying attention to vertices, it is important that children make note of the sides. Consider the following figures (see Figure 7):



*7a: Concave sides* *7b: Convex side* *7c: Straight sides*

*Figure 7. Illustrating straight and curved sides.*

Once again, all three figures have the same prototypical orientation with a point centered on the top. They all have a quality of "threeness". All three figures have three "points" or "corners". Yet, only the triangle has three vertices. What distinguishes between points and vertices is their connection to sides. Children may not always be aware of the distinction between points and vertices and may therefore have difficulty identifying the first two figures as non-triangles if they only focus on the points. For example, C5, a five year old, claimed that all of the above figures were triangles and accompanied each of the three identifications with the same reason, "It's a triangle because it has three corners." Of course, some children, operating at the first van Hiele level, will claim that the first two figures are triangles because "they look like triangles." However, this is not what C5 claims. He takes notice of what he terms corners. He also mentions the critical attribute of three. The question is: What is C5 missing? We may surmise that C5 is cognizant of the necessity of vertices, despite the terminology used. What he fails to notice is the curved sides in Figures 7a and 7b. In fact, C5 does not mention sides at all, either curved or straight. Having a working definition means that the child is able to check the figure not only for the existence of vertices but also for sides. Taking into consideration the young age of the children means realizing that children may notice attributes of a figure but may not necessarily see the relationship between these attributes. A working definition allows the child to run through a check list of attributes, where each attribute must be accounted for. If C5 had made use of such a definition, then, in addition to noticing three points, the sides would have been checked. Of course, it also must be understood that the sides must be straight.

If the term vertex must obviously be taught to children, it is not so obvious that children must learn the meaning of "straightness". Yet, this too should not be taken for granted. Some children will ignore the curvature and claim that the first two figures are triangles because "they have three sides" or because "they have three lines." Yet, children who participated in our programs were able to identify the first two figures as non-triangles, noting that not all of the sides were straight. As one four-year old reasoned regarding Figure 7a, "It's like a triangle, but it isn't. It has three vertices but three sides that are curved. They need to be straight." We can almost picture the child going through a checklist in her mind. She notes the visual similarity to a triangle, notes the three vertices, and still correctly identifies this figure as a nonexample because the sides are not straight. In a different case, the researcher presented the first figure to a three year old participating in our program and the following discussion ensued:

R: Look. I found a triangle.
C6: No. (Smiling) It's not a triangle.
R: No? I made a mistake? Why isn't it a triangle?
C6: Because it's curved!

The child then went on to pick up a different figure which was indeed a triangle and handed it to the researcher. This episode is interesting for two reasons. First, the child was confidant enough in her knowledge to disagree with the grown up authority. In addition, this three-year old child did not state that the line "was not straight". Instead, she had learned that the opposite of straight (in geometry) is curved and used this word to explain to the researcher why the researcher had "made a mistake". Teaching children correct mathematical language may serve as tool that can then be used when reasoning about mathematics.

Before moving on to the next critical attribute we pause to consider the difference between Figures 7a and 7b. These figures differ in two ways. The first has three curved sides and the second has only one. The first is curved inward and the second outward. In our research, we have come to understand that when a critical attribute is tampered with, depending on the type of tampering, children's reactions may differ. For example, two four-year olds who had not participated in our programs incorrectly identified Figure 7a as a triangle. The first gave no reason at all. The second merely added the word "three" and gestured toward each of the points. Yet, although these children incorrectly identified Figure 7a, they both correctly identified Figure 7b as a nonexampe. Again, the first did not say anything. However, he did gesture towards the curved line. The second verbalized that the line was curved. Did they not see that the first figure also had curved lines? Although we do not explicitly have the answer to this question, it could be that for the children there is a difference between concave and convex lines. It could also be that when all three lines are the same, it is not as apparent that something has gone amiss as when two lines are the same and one is different.

Consider now the critical attribute of closure. Referring back to Table 1 we note that the pre-kindergarten children tended to ignore this attribute more than the kindergarten children. In Figure 8 we present possible ways of presenting this attribute to children.

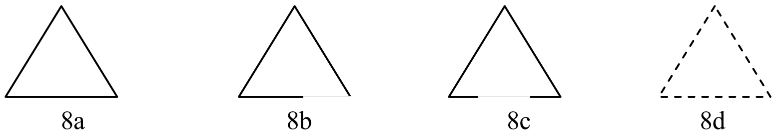



*Figure 8. Illustrating closed and open figures.*

Figures 8b, 8c, and 8d are all open figures. Take a minute to contemplate the difference between how these figures are open and how each one may affect a child's understanding of the attribute of being closed. Figures 8b and 8d were presented to 107 4-6 year olds not participating in our programs. While approximately 75% of the children correctly identified Figure 8b as a nonexample, only 7% of the children identified Figure 8d as a nonexample. Why such a huge difference? Several possible reasons may explain these results. First, children often engage in activities that involve "connecting the dots" to form some picture. Such an activity is often used when teaching young children to write letters or number symbols. Thus, when presented with a dotted figure, they may automatically assume that the dots are to be connected. Another explanation for this phenomenon may be the result of Gestalt recognition of figures, appropriate for young children operating at the first van Hiele level of geometrical thinking. Recall, children at this level take in the whole, without regard for the attributes. It may also be a combination of these reasons. As Fischbein (1993b) noted, Gestalt features may be inspired by practice. The child may know that the dotted lines means that the figure is not closed, but the practice of connecting dotted lines may lead him to Gestalt thinking.

The difference between Figures 8b and 8c lies in where the figure is broken. How could this make a difference? In our study of 65 kindergarten children (Tsamir, Tirosh, & Levenson, 2008), we noted that 20% of the children claimed that a figure similar to that of Figure 8b (see the open "triangle" in Figure 4) was not a triangle because it only had two vertices. One three-year old claimed that this figure was not a triangle because "there is no vertex here." In other words, Figure 8b may be considered a non-triangle because it has a missing vertex, thus violating the critical attribute of threeness. We do not claim that this reason is inappropriate. Only that, if we specifically want the children to focus on the attribute of being closed, then Figure 8b may not be enough.

We end with the critical attribute of a triangle having three, and only three, vertices and sides. On the one hand, it may seem that this critical attribute is obvious to children. Our research with kindergarten children (Tsamir, Tirosh, & Levenson, 2008) suggested that for a triangle, the perception of threeness has a stronger pull than the necessity for it to be closed or for its vertices to be pointy. Furthermore, it might be argued that the bond between a triangle and its attribute of threeness is also expressed in the name itself, which in many languages, including Hebrew, stems from the root three. So, if a child perceives threeness in a shape, then the child sees a triangle. Conversely, a shape which is missing threeness cannot be a triangle.

If the necessity for threeness is obvious to children, it may be even more difficult to bring the criticalness of this attribute to children's' attention. How may we demonstrate this attribute to children?

Consider the following figures:

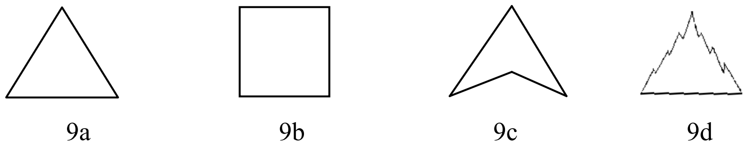



*Figure 9. Illustrating the attribute of threeness.*

Which of the above demonstrates how the critical attribute of three may be violated? We have already seen that many young children correctly identify the square as not being a triangle, simply by naming it a square. Thus, although the square is not a triangle because it has four, and not three, vertices and sides, it may not be the best choice for bringing this critical attribute to light. On the other hand, Figure 9c, like the square, is also a quadrilateral with four sides and vertices, but unlike the square, most young children are not familiar with the name quadrilateral. Therefore, it would be unlikely for children to discount this figure as a triangle, because they can name it otherwise. In addition, Figure 9c resembles the prototypical triangle, in that it has a vertex centered at the top. It also gives the illusion of threeness while having four sides. Figure 9d also gives the illusion of threeness. Zooming out, the overall picture one perceives is that of a three-sided figure of which two sides are jagged, losing the critical attribute of straightness. If one zooms in on the non-horizontal sides, then the correct definition of this figure would be a 15-sided polygon, thus losing the critical attribute of threeness. As with the other critical attributes, there is no "best" way to illustrate the violation of an attribute. Instead, the teacher and researcher should be aware of the possible issues which each example and nonexample may elicit from the child.

Although our study with kindergarten children suggested that the attribute of threeness was intuitively connected to triangles, this may not be true of younger children. Children between the ages of 3 to 6 are still developing their counting skills, including one-to-one correspondence and cardinality. In our programs, we offer children many opportunities to count objects. The three-year olds often need help with one-to-one correspondence. And even when they have mastered one-to-one correspondence, they have not necessarily reached the understanding of cardinality. When asked how many vertices the triangle has, many three-year old children will correctly count the vertices and repeat the process of counting for each time they are asked about the number of vertices (or sides). Recall that Dan, the three-year old quoted in the beginning of this chapter, mentioned that the first shape was a triangle because it had vertices. He did not mention that there were three vertices, although Dan had pointed to each vertex. The attribute of threeness, as opposed to the other attributes, is connected with the child's development of counting skills. Until he has mastered this skill, he may not be ready to consider this attribute. That is not to say that counting the vertices is inappropriate for three-year olds or that the critical attribute of threeness should be omitted for the youngest of children. We take the stand that children may increase their mathematical skills with proper instruction. Counting vertices, pointing out the difference between the three vertices of a triangle and the four vertices of a square, may increase children's awareness of the connection between number and geometry skills. This is important for all ages.

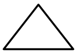
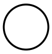
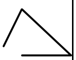
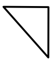
## 2.4 SUMMARY

Let us consider one more time the three children quoted at the beginning of this chapter. Dan, age 3, notices the vertices of the figures. He does not mention any of the other attributes of a triangle. Nancy, age 4, mentions vertices and sides. Jordan, age 5, mentions vertices, sides, and that the figure is closed. It would be too simplistic to conclude that the age of the child determines to what degree he or she is capable of working with a concept definition. Children develop at different paces and it is our responsibility as educators to help each child move forward. On the other hand, as pointed out above, some attributes, such as threeness, may be linked to development. In general, young children can learn to incorporate the concept definition as a tool for identifying examples and nonexamples of triangles and thus increase their example and nonexample space. In a more recent study, we investigated kindergarten children's identification of various examples and nonexamples of triangles. Of the 215 participants, 134 had learned in kindergartens which participated in our program and 81 did not. Results are presented in Table 3. These results demonstrate that children learning in our program identify correctly more examples and nonexamples than children not in our program. Furthermore, over 90% of the reasons given by children learning in our program were based on the critical attributes of the triangle.

We would like to remind the reader that our goal is for children to use the definition of a triangle, not a formal definition, but a working definition tailored to meet the developmental needs of the child, as the decisive criterion for determining if an object is or is not an example of a triangle. We propose that a minimal definition is not only insufficient for young children but may actually be confusing to young children. Instead, we propose developing working definitions which bring to the fore each of the critical attributes. By illustrating a variety of examples and nonexamples we explored how children may develop an appreciation for each of the different critical attributes and how reasoning goes hand in hand with identification.

Not all geometrical figures are easily defined in preschool. In the next section, we consider the case of circles and how developing children's conception of a circle may differ from developing children's conception of triangles.

*Table 3. Frequency of correct identification and critical attribute reasoning among kindergarten children.*

| | Correct Identification | | Critical attribute reasoning | |
|---|---|---|---|---|
| | Non-program children (N=81) | Program children (N=134) | Non-program children (N=81) | Program children (N=134) |
| Equilateral triangle  | 100 | 100 | 40 | 98 |
| Circle  | 100 | 100 | 30 | 90 |
| Rounded "triangle"  | 19 | 93 | 42 | 98 |
| Zig-zag "triangle"  | 64 | 99 | 30 | 94 |
| Scalene triangle  | 17 | 95 | 11 | 96 |
| Pentagon  | 68 | 90 | 17 | 96 |
| Open "triangle"  | 81 | 100 | 59 | 100 |
| Concave "triangle"  | 53 | 97 | 35 | 99 |
| Right triangle  | 65 | 99 | 21 | 95 |

CHAPTER 3

# THE CASE OF CIRCLES – WHEN THE CONCEPT DEFINITION IS INAPPROPRIATE FOR THE AGE OF THE CHILDREN

It is obvious that mathematically, triangles and circles are different. But are they different psychologically? Let's try an experiment. Draw a circle. Now draw another circle. Now draw another circle. In what ways are the circles that you drew different? Perhaps they are different sizes. Perhaps they are different colors. However, the symmetry of the circle does not allow for many different types of variations. Unlike triangles, non-critical attributes such as orientation or aspect ratio do not apply to circles. The circle is also one of the most readily identifiable figures among young children. In one study, children ages 4-6 years old identified correctly more circles than they did squares, triangles, and rectangles (Clements, Swaminathan, Hannibal, & Sarama, 1999).

Let us try another activity. For each figure below (see Figure 9), tell if the figure is or is not a circle and why. Were you able to explain why figures c and i are circles? Were you able to explain why the other figures are not circles? How is this activity different from a similar activity involving triangles? (Perhaps take a moment to look back at some examples and nonexamples of triangles as in Chapter 2, Figure 4 – Intuitive and non-intuitive examples and nonexamples of triangles.) How might children respond to this circle activity?

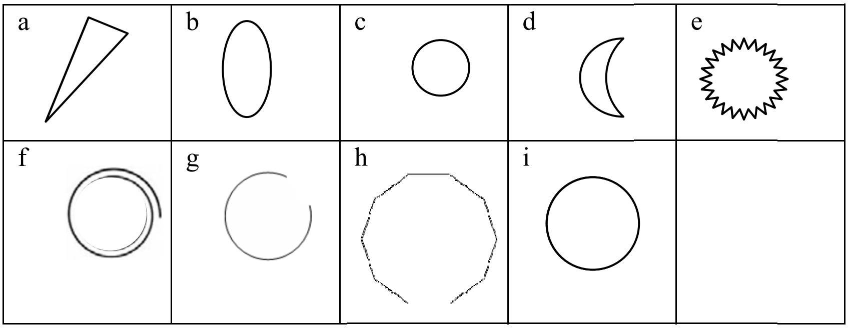



*Figure 1. Is it a circle? Why?*

Amit, Dan, and Yael are three four-old year olds who have already learned how to differentiate between triangles and non-triangles using critical attribute reasoning. All three children know that figures c and i are circles. When asked to explain why figure c is a circle each responds differently:

Amit: It's a circle because it doesn't have any vertices.
Dan: It's a circle because... you see it's a circle.
Yael: It's a circle because it's closed.

Amit and Yael refer to attributes they had previously learned in connection with triangles. Amit's response indicates that he knows that a circle may not have vertices. Perhaps he is trying to express how the circle and triangle are different. Yael, on the other hand, points to an attribute that the circle has in common with the triangle. Both the triangle and the circle are closed figures.

If a figure does not have vertices must it then be a circle? According to Amit's explanation, figure b, for example, would also be a circle. Yet, Amit claims that figure b is not a circle because "it's like an egg." Yael's explanation is also not satisfactory. Yes, a circle is closed but so are many figures than are not circles. Being closed is not enough reason for a figure to be a circle. Like Amit, Yael claims that the ellipse, also a closed figure, is not a circle, because it is like an egg.

Dan cannot explain why figure c is a circle. It seems that for Dan, he just sees the circle. This is typical of Gestalt reasoning, where the object appears first as a whole and only after does one perceive parts as opposed to first seeing parts which come together to form a whole.

As we discussed in the previous section, the mathematical definition of a concept is the decisive criterion by which we sort instances into examples and nonexamples of that concept. A circle is defined as the locus of points in a plane which are equidistant from a given point called the center. And therein lies the problem. How can this definition be made "user friendly" to young children? The notion of a collection of points which come together to form a closed figure is quite abstract. In addition, we would need to explain the notion of distance. One way of illustrating these ideas to children would be to physically demonstrate how a circle may be constructed. We could build a compass with the children by tying one end of a string to a pencil or some other marker and the other end to an object with a sharp point such as a thumb-tack. By placing the thumb-tack on the paper (and keeping it still), we could then extend the compass string fully, pull the string around the center thumb-tack and draw the line that it subscribes. While this demonstration may illustrate the notion of a center and distance from the center, translating the physical motions to a verbal definition for such young children, would be rather difficult. It is no wonder, then, that the children could not satisfactorily explain why figure c is a circle.

The circle is a classic figure which is perceived in its entirety. One does not look at the "parts" of the circle and put them together. The center is often not even marked. The only variance between examples of circles is their radii, the distance between the center and the points on the circle. Thus, if we talked about intuitive and nonintuitve examples of triangles, this dimension, would not apply to circles.

On the other hand, among non-circles, there is a wide variety. Look back at the non-circles in Figure 9. How do they differ from each other? Are some intuitive non-examples of circles, immediately recognized as such by children?

We have already seen that when a child can name a figure, he places it into one category, simultaneously excluding it from another category. Thus, the triangle may be considered an intuitive nonexample of a circle. In addition, figures d, e, and g (a “crescent”, a “sun”, and an open “circle” respectively) were readily identified as non-circles. Regarding the crescent, children either pointed to the two points or simply claimed that the figure was a “moon” and therefore was not a circle. This second reason may be problematic. At times, the moon may appear as a full circle and not as a crescernt. In general, although we have not defined for children what a circle is, we may still strive for them to recognize and name attributes which are contradictive to circles. This indeed was the case for the open “circle” – all three children pointed to its openness when stating that it was not a circle. Regarding the sun, children either pointed to the many vertices or the lines.

When explaining why the triangle was not a circle, two of the children said it was not a circle, “because it has vertices.” However, Yael claimed, “It is not a circle because it has three vertices and three sides and … but it’s closed.” Yael is referring to her working definition of a triangle. In essence, she is explaining why the figure is a triangle, which is not the same as explaining why the figure is not a circle. It could be that Yael, by explaining why this figure is a triangle, is claiming that if a figure is one shape then it may not be another. This is problematic when it comes to, for example, a square, which is not only a square but also a rectangle. Thus, claiming that a figure is one shape does not exclude it also from being another shape. Knowing that the shape is a triangle is different from knowing that a shape with vertices and sides cannot be a circle.

What is the difference between Yael’s explanation and that of the other two children? Just as it is important to teach children that an example must uphold *all* of the critical attributes of that concept it is important to teach them that if even only *one* of the critical attributes of that concept is not upheld, the figure is not an example of that concept. In other words, if a figure has any vertices at all, then it is not a circle. It does not matter how many vertices it has. Notice how Yael pauses before she comments that the triangle is closed. She began by contrasting the triangle with the circle. The triangle has three vertices and three sides. It is possible that she pauses as she realizes that although the triangle is closed so is the circle. Does she realize at that point that her explanation was not focused on why the figure was not a circle but rather on why the figure is a triangle? Perhaps, for a moment, she is not sure if the figures are not the same; after all, they do have something in common.

Although the decagon and triangle are both polygons, none of the children claimed that the triangle was a circle while a few did claim that the decagon was a circle. Perhaps, as with the “bonfire” triangle-like figure in the previous section, children zoomed out, merging the sides of the decagon into a continuous curved line. Most children could not explain why they claimed the decagon was a circle. Pam, on the other hand, explained, “it’s like when children join hands in a circle.”

The most challenging figure for the children was the spiral. Regarding the spiral, children’s identifications as well as reasoning were mixed:

Roy: It is a circle. (He traces the figure with his finger.)
Molly: It is a circle but it's open. It's not a circle because it's open.
Gila: It's not a circle because it's not closed.
Jake: It is a circle because it's round.

Roy's explanation consists of tracing the figure with his finger. Perhaps he, like Jake, is focusing on the roundness of the spiral. Jake states that the spiral is a circle because it's round. However, roundness is not a term well-defined. The ellipse may also be considered round. Gila immediately notices that this figure is not closed and thus cannot be a circle. Molly, on the other hand, fluctuates. At first she claims that the spiral is a circle even thought it is open. However, on second thought, she concludes that if the figure is open, it may not be a circle. We would conclude that the spiral is a non-intuitive non-circle.

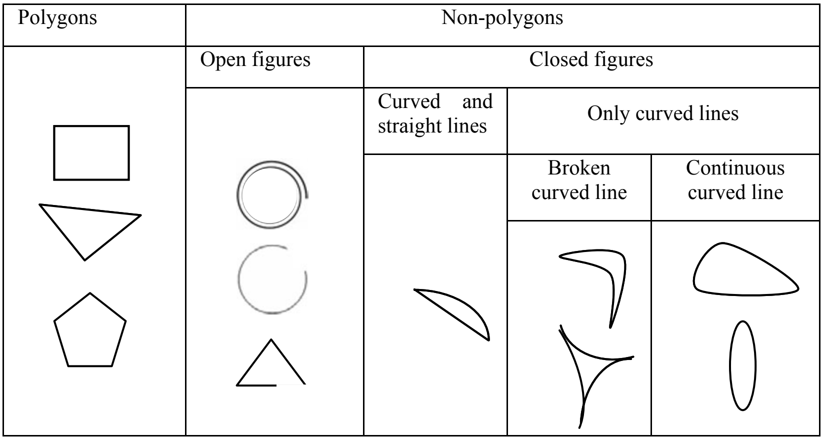



*Figure 2. One possible way to categorize non-circles.*

In general, when considering nonexamples of circles, we suggest the following way to sort out the many varieties (see Figure 2). We begin by differentiating between polygons and non-polygons. Polygons cannot be circles because by definition they have sides and vertices, which are not attributes of the circle. Among non-polygons, we may separate between open and closed figures. Open figures may not be circles. Among closed figures, if the figure is made up of both curved and straight lines, it cannot be a circle. Thus far, we have referred to attributes which the children may recognize from learning about triangles. We now move on to attributes less familiar to children who had only learnt about triangles and other polygons. Closed figures may consist of one continuous curved line or they may consist of a broken curved line. A circle must consist of a continuous curved line. Finally, among closed figures consisting of one continuous curved line, non-circles are those which do not have a center point which is equidistant from all points on the curve. It is the last two attributes which we believe that children at this age may not grasp.

When we had previously discussed learning and teaching triangles, we sorted figures along two dimensions: a psycho-didactical dimension, as well as a mathematical dimension. Above, our categorization of non-circles was made only along mathematical dimensions. It remains a question as to which non-circles might be intuitively recognized as such. If children have already learned about and know the names of various polygons, then those polygons may be intuitively recognized as non-examples of circles. On the other hand, if children have only learned about triangles and squares, then a decagon, for example, may not be an intuitive nonexample of a circle as the multiple lines may begin to resemble a curve.

Not being able to form a verbal working definition of a circle for children leads to difficulties which were not encountered with triangles, pentagons, and hexagons. If the child cannot refer to a working definition then it is almost impossible for him or her to explain why a figure is a circle. Among nonexamples, some may be explained by pointing to an attribute which contradicts the existence of a circle but many cannot be explained in this manner. This dilemma led us to believe that in the case of circles, we should focus on strengthening the concept image of circles without referring to any definition.

# THINKING ABOUT OTHER SHAPES

In concluding this part, we ask you to consider how the section on developing children's conception of triangles may be applied to developing children's conception of pentagons or hexagons. What might be an appropriate working definition of a pentagon? What examples of pentagons would you present to children and which nonexamples would you present? In our work with children we most often begin with introducing children to the concept of a triangle. We have found that children are responsive to our efforts and enjoy "playing" with and learning about triangles, a figure which seems to be familiar to them. It is only after children have come to accept that some attributes of a triangle are critical and others are not, that we begin to work on pentagons. As Ken, a five-year old boy commented when describing the pentagon, "it's like a triangle but is has five."

You may have noticed that quadrilaterals were not discussed in this chapter. In the beginning of this part of the book we reviewed theories of children's development of geometrical concepts and reasoning including the difficulties which may arise along the way. One of these difficulties includes grasping a hierarchical structure. Young children may find it hard to recognize that a square belongs to the group of rectangles which in turn belongs to the group of parallelograms which in turns belongs to the group of quadrilaterals. Complicating matters even further is the issue of naming. How can one shape have two different names? These questions did not arise when discussing other polygons. A triangle with equal sides and equal angles is called an equilateral triangle. For one, the name "equilateral triangle" evokes the category to which it belongs; it consists of two words, an adjective which is combined to the noun it is describing. Thus, even if we were to introduce children to the term equilateral triangle, it would be less confusing than introducing a totally new name for this figure. In addition, a definition of a rectangle or a square would have to include concepts such as equal distance or angles, which may be very complicated for young children to grasp. Perhaps, we may encourage children to widen their concept image of squares and rectangles to include those with various orientations without relating specifically to the concept definitions. We are left with many questions regarding quadrilaterals.

After considering how geometrical concepts may be developed among young children we are ready to consider how theories reviewed in this chapter may be applied when actually engaging children with geometrical tasks. This is the focus of the next part of this book.

PART 2

# ENGAGING YOUNG CHILDREN WITH GEOMETRICAL TASKS

Consider the following scenario: Rita, the kindergarten teacher, places in a bin several cutouts of many different shapes – triangles, squares, circles, pentagons, hexagons – in a variety of colors and sizes. She assigns two five-year olds the task of pulling out all of the triangles. In a different corner of the classroom, several children are working individually on tracing the outline of different geometric shapes. In the blocks corner, a group of children are building castles. Many of the blocks are in fact three-dimensional geometrical figures such as cubes and cylinders. What are the differences between these activities? What are the similarities? What knowledge related to geometry may the children be learning from the different tasks in which they are involved?

Part One of this book was dedicated to studying preschool children's development of geometrical concepts. While children may certainly acquire geometric knowledge from their everyday lives, with support and instruction, geometric development may progress further (van Hiele, 1958). What kinds of tasks may we implement with kindergarten children in order to help them along with this development? Consider again the three activities described above. As children engage in the above tasks, what messages may they be receiving regarding geometry in particular and mathematics in general?

Students spend much of their time in the kindergarten classroom, and later on in the mathematics classroom, working on tasks. These tasks often mediate between the teacher and students, conveying "messages about what mathematics is and what doing mathematics entails" (Styliandes & Styliandes, 2008, p. 859). In Part Two of this book, we focus on geometrical tasks implemented with young children. The first chapter offers theoretical background regarding mathematical tasks in general and geometrical tasks specifically. In the second chapter we describe two geometrical tasks that may be implemented with children in preschool, possible scenarios that may occur when implementing these tasks, and results from studies in which children engaged in these tasks.

The third chapter in this part presents a variety of tasks which teachers reported implementing in their preschool classes.

CHAPTER 4

# MATHEMATICAL AND GEOMETRICAL TASKS: THEORIES AND RESEARCH

As we stated in the beginning of this book, there is a need at the beginning of any dialogue to establish a common language and a common background between participants. This chapter intends to fulfill that purpose by providing the terminology and theory on which the others chapters in this part of the book rest. It begins by taking a look at academic tasks in general and moves on to mathematical tasks in particular. What are the elements of a task? What principles guide their design? How are they implemented in the mathematics classroom? We then review curricular guidelines and research related to geometrical tasks. What kinds of tasks are suggested by various guidelines and what kinds of tasks have been implemented in the past? What were the aims of these tasks? Which of these tasks were implemented with young children?

## 4.1 ACADEMIC AND MATHEMATICAL TASKS

Our discussion of mathematical tasks begins with a general analysis of academic tasks. From various literature regarding academic tasks, we surmise that academic tasks are those tasks implemented in a classroom setting with the aim of promoting subject matter knowledge among the students. In this book our setting is the kindergarten classroom. While children most certainly learn a great deal from periods of free play, in this book we focus on the tasks that kindergarten teachers implement with the children in their class in order to promote particular knowledge. Thus, the first task described in the introduction above, pulling out all of the triangles from a bin containing a variety of figures, may be considered an academic task. As we continue to discuss academic and, later on, mathematical tasks, it may help to keep this task in mind.

What comprises an academic task? According to Doyle (1983, 1988), academic tasks are composed of four interdependent elements: products, resources, operations, and accountability. In general, students are expected to generate a product (solution to a problem, oral responses in class) by using the resources made available by the task or problem (textbooks, notes, worked out examples) and by operating with these resources (remembering previous lessons, applying appropriate rules, formulating algorithms) taking into consideration accountability of the task (importance of the task in the general classroom scheme). What would be the product of the "pull out all the triangles" task? The intended end product is a pile of cutout triangles and a bin full of figures that does not contain a single triangle. What resources do the children have to work with? The answer to this question may depend on the point in time that this task is offered. For example, if the teacher gives this task as an introduction to learning triangles, then the resources available to the child might stem from outside of the kindergarten, for example from his home environment. On the other hand, if the teacher has already introduced triangles to the children, and if the teacher has hung up in the classroom drawings of various triangles, then the children working on this task might well refer to other known triangles as they solve the task at hand.


*Figure 1. (Translated from the Hebrew) Geometry in our kindergarten: Quadrilaterals.*

How does the child operate with these resources? Perhaps, as discussed in the first part of this book, the child has learned an appropriate working definition for a triangle that he or she may refer to. A definition such as, a triangle is a closed figure with three pointy vertices and three straight sides, may serve as a resource for checking the various cutouts in order to discern if a cutout meets the necessary criteria. Regarding the importance of the task in the general classroom scheme, this is very much up to the kindergarten teacher. Will the teacher check to see if the task was completed satisfactorily? Will the children implementing the task feel that they have accomplished an important assignment?

Doyle's decomposition of academic tasks has been used in several studies to analyze mathematical tasks. For example, Stein, Grover and Henningsen (1996) referred to Doyle's task analysis when describing mathematical tasks used in reform classrooms. However, unlike Doyle who referred to short succinct tasks, their conception of a mathematical task encompassed a longer period of duration, such that the student was able to focus and develop a specific mathematical idea. A task was said to continue as long as the underlying mathematical idea did not change. Thus, if the task of pulling out all of the triangles from a bin is followed by explaining why each of the cutouts is indeed a triangle, and perhaps this is followed by tracing the cutout triangles, then the task only begins with pulling out triangles and doesn't end until the child has traced each triangle. The task may even continue on if the child then continues to color in her drawing of the triangle.

Herbst (2003) adapted Doyle's four task components to investigate a teacher's management of the use of novel tasks in teaching mathematics. The four components were used to investigate: (a) the product expected of the students as indicated by the teacher, (b) the representations given as resources, (c) the conceptual actions used by the students, and (d) the way in which work on the task related to the customary obligations of the teacher and students. Again, we may relate what Herbst (2003) investigated to our kindergarten triangles task. Herbst considered different representations which served as resources. Although different representations of a concept may not necessarily be the same thing as different examples of a concept, we may consider different examples of triangles as resources, such as prototypical triangles, triangles with different orientations, and triangles which are non-intuitive. We may also consider as a resource nonexamples of triangles and then discern between intuitive and nonintutive nonexamples (as discussed in the first part of this book). Stylianides and Stylianides (2008) used Doyle's decomposition to study the implementation of mathematical tasks embedded in real-life contexts by comparing these elements at different points of the implementation. We might relate this to our "pull out the triangles" task by asking the following question: what resources does the child turn to at different points of implementation? To begin with, the child may use everyday contexts as a resource and thus pull out all triangles which resemble "the roof of a house". After this phase is completed, other resources may be called upon. Are there drawing of triangles in the near vicinity? Will the child turn to the teacher or to other children for help? At what point?

Up until this point, we have only referred to the "pull out the triangle" task. However, we may begin to ponder the other tasks mentioned in the beginning of this chapter. What types of blocks are available in the block corner? What types of shapes are the children asked to trace? These shapes and blocks may serve as resources as children learn about geometrical figures. What would be the product of building with blocks? Does the activity end with a tall tower? A different product might be produced depending on if the child chooses to build only with cylinders or only with cubes. We continue to refer to these tasks in the following sections.

As shown above, mathematical tasks may have similar components to other academic tasks. Then what differentiates mathematical tasks from academic tasks? Is it only the mathematical context of the task? In the next section we focus on the uniqueness of mathematical tasks, beginning with their design.

## 4.2 DESIGNING MATHEMATICAL TASKS

Bettina designed a group task for the children in her kindergarten class related to three-dimensional solids. In advance she prepared an assortment of geometric solids consisting of two cylinders taken from the set of blocks in the block corner, two different size balls, three cubes made from different materials, a hard plastic 4-sided pyramid, and a hard plastic cone. Without the children seeing, she placed one of these solids inside a brown paper bag and each child, in turn, was to place his or her hand in the bag, feel the solid, and tell which one was in the bag.

What were Bettina's considerations when she was designing this task? If you were assigned to implement this task with kindergarten children, how would you interpret the task? The first step when designing a mathematical task is to differentiate between the actual task and the mathematical activity which it is meant to promote. In the above example, the task is to come up with the name of the solid hiding inside the bag. The mathematical activity, however, concerns more than identifying a solid by name. It includes discerning critical attributes of solids such as the number of faces or the absence of any base. Watson and Mason (2007) differentiated between "the task as conceived by the author, as interpreted and intended by the teacher (if she is not the author), and as interpreted and constructed by the learner" (p. 206). That is, the teacher presents a task to the students that she may or may not have designed, sets the rules for engagement, and has in mind a set of goals which are to be achieved by the students engaging in the task. Mathematical activity is what (hopefully) occurs as students engage in the task. This activity may take the form of working on the assigned task individually or discussing issues that arise from the task with the teacher or with other students.

Watson and Mason (2007) further refined this idea by referring to the explicit or outer task as the specified or undertaken task whereas the implicit task is "what it affords of mathematical themes, concepts, theorems, connections to other topics and techniques, multiplicity of approaches, interpretations and representations" (p. 206). The task described above affords the teacher to discuss with the children general geometrical themes such as the necessity to use critical attributes when identifying examples of geometric figures. It affords the teacher the opportunity to relate three-dimensional solids to two-dimensional shapes and clarify the differences. It also affords the opportunity for children to realize that identifying a figure is not dependent on how it "stands". That is, even if the cone is not standing on its base, if it is in a bag and can be rotated in different directions, it remains a cone.

In addition to the affordances of a task, the designer must also consider its constraints, how the task may limit wider possibilities. What may be lost by placing the solids in a paper bag? What may be given up by not including non-geometric solids in this task? This is not to say that constraints are necessarily a negative. Constraints may serve to focus the student on the mathematics intended by the teacher, thus enhancing the process of concept formation (Watson & Sullivan, 2008). The affordances and constraints of a task may also be related to the interactional norms that have been developed between the teacher and students regarding engagement with tasks.

It is important to recall that the student is unaware of the implicit task. The student is also less knowledgeable of the mathematics than the task designer or teacher. Thus, the task designer must be aware of two perspectives: that of the observers' point of view and that of the actor's point of view (Figueiredo, van Galen, & Gravemeijer, 2009). The observer's point of view is that of the designer or teacher who, besides knowing the mathematics inherent to the task, has an understanding of the context and its limitations. Yet, the designer must place him or herself in the role of the student. This is the actor's point of view. Taking this point of view requires the designer to anticipate the different ways a student may interpret the task, which may be very different than the way it was intended to be interpreted. For example, when working on mathematical problems situated in real-life contexts, what is real for one student may not be for another. In our case, a child who associates a cone with an ice-cream cone or a clown's hat may perceive cones as being hollow receptacles which can be filled up. It is up to the designer, as well as the teacher implementing the task, to take this into consideration. It also means taking into account students' current knowledge and the hypothetical learning processes of the student (Simon & Tzur, 2004). Have children learned about triangles before engaging in a task which includes pyramids? In this light, mathematical tasks become tools for promoting the learning of mathematical concepts and are thus a key part of instruction.

Task designs are often based on different ways of conceptualizing mathematics. For example, mathematics may be viewed as a complex, but stable, set of ideas and theories which students come to understand by learning different mathematical topics. On the other hand, students bring to the classroom a variety of experiences related to mathematics which introduces creativity as well as spontaneity to the classroom. Krainer (1993) described this dilemma in mathematics instruction as the security versus insecurity dilemma. This dilemma gives rise to the need for tasks, including tasks in preschool, which will provide a secure path to the learning of concepts as well as less secure tasks which allow for investigation and discovery on the part of the student. Tasks which provide both security and insecurity may be called powerful tasks. They are characterized by two pairs of properties:

> 1a) Team spirit: this property means that tasks should be well interconnected with other tasks. The "horizontal" connection of tasks can be seen as a contribution to the security of mathematics course.
> b) Self-dynamics: This property means that tasks facilitate the generation of further interesting questions. The "vertical" extension of tasks to open situations can be seen as a contribution to the insecurity of mathematics course.
> 2a) High level of acting: This property refers to the initiation of active processes of concept formation which are accompanied by relevant ("concept generating") actions.
> b) High level of reflecting: This property implies that acting and reflecting should always be seen as closely linked. An important aspect of reflection refers to further questions from the learners (which in their turn could lead to new actions). (Krainer, 1993, p. 68)

Taken together, tasks with these properties embody the security versus insecurity dilemma combining acting with reflecting allowing students to relate new ideas to other concepts.

Do geometrical tasks in the kindergarten also reflect the security versus insecurity dilemma? The children in Nadine's kindergarten have learned to distinguish triangles from non-triangles by checking if the figure is closed, has three pointy vertices, and three straight lines. The children have also been introduced to non-square quadrilaterals by comparing the attributes of these quadrilaterals to triangles. Like the triangle, the quadrilateral is closed, has pointy vertices and straight lines but instead of three vertices and three lines it has four vertices and four lines. One child is then given the following task: From an assortment of geometrical figures, including triangles, non-square quadrilaterals, squares, pentagons, hexagons, and circles, pull out all of the quadrilaterals. What elements of this task give the child a secure feeling? How is this task connected to previous mathematical concepts learned in the kindergarten? What elements give the child an insecure feeling? What new questions may result from this task? On the one hand, the child has practiced and feels comfortable with the process of counting vertices and sides and identifying closed figures based on this count. This gives the child a secure feeling. On the other hand, will the child accept a square as a quadrilateral? The concept image of a square is often formed before children reach kindergarten (Clements, Swaminathan, Hannibal, & Sarama, 1999) and thus may not trigger the child to count its vertices and squares. Furthermore, children who can name the square may not accept that it can have a second name and that it also belongs to the category of quadrilaterals (Waxman, 1999). Thus, this activity requires a high level of reflecting and includes elements of insecurity.

The principles of task design are not only based on how one conceptualizes mathematics. Swan (2007) claimed that designers also take into consideration the values and purposes of learning mathematics and the mechanisms by which that learning takes place. His own task designs were based on five distinct purposes for learning mathematics: developing fluency when recalling facts and performing skills, interpreting concepts and representations, developing strategies for investigation and problem solving, awareness of the nature and value of the educational system, and an appreciation of the power of mathematics in society. He also incorporated into his tasks a social constructivist theory of learning (Swan, 2008). In this spirit, we may ask ourselves: What is the purpose of learning geometry in general and what is the purpose of learning geometry in kindergarten? If one of our answers to these questions relates to developing spatial sense, then we may provide tasks in which children need to manipulate and rotate geometrical figures to solve a puzzle. If another purpose of learning geometry in kindergarten is to develop children's deductive reasoning skills and to help children progress along the van Hiele levels of geometric reasoning, then we may design tasks where children are encouraged to see the relationships between different attributes of a geometrical figure.

Swan (2008) designed five task 'types' that encourage concept development by promoting distinct ways of thinking and learning: classifying mathematical objects, interpreting multiple representation, evaluating mathematical statements, creating problems, and analyzing reasoning and solutions. For example, a task which entails classifying mathematical objects may include three algebraic expressions where the student is asked to write why each expression may be considered different from the other two. Similarly, in the “pull out all the triangles” task, children could be encouraged to first explain how the cutouts they pulled out from the bin are different from those they left in the bin. Children could then be encouraged to classify the pulled out triangles in different ways. Other children may then be requested to evaluate their friend’s classifications of the figures.

Designing tasks that meet the needs of conceptually oriented instruction was also discussed by Silver, Mesa, Morris, Star, and Benken (2009). In order to describe tasks that encouraged conceptual development they focused on the task’s mathematical and pedagogical features. Mathematical features include not only the mathematical topics of the task but the mathematical cognitive demands inherent in the task. Low cognitive demand tasks require students to recall and remember facts or use routine applications of known procedures. In the kindergarten, this may consist of having children repeat again and again the critical attributes of a triangle (not that this is an activity which we recommend). High cognitive demand tasks require students to analyze, create, or evaluate facts, procedures, and concepts or to engage in metacognitive activity. In the kindergarten, we may consider a task which encourages students to raise their own geometrical conjectures or evaluate their friends’ conjectures and justifications. For example, in the “guess which solid is hiding in the paper bag” task, a child may not only say out loud which solid he thinks is in the bag but may describe what he feels to his friends in the group and explain why the solid is, for example, a cone, and not a cylinder.

The pedagogical features of a task refer to the organization and enactment of the task. For example, tasks which are designed to encourage collaboration and discourse among several students might be considered a pedagogical feature of a task that has the potential to encourage conceptually oriented learning. Other examples of pedagogical features that support such learning are tasks which call for the application of mathematics to other contexts, thereby connecting mathematics to the real world and tasks which call for using hands-on or technological tools. Pedagogically, we may also consider designing tasks that take advantage of geometric software to encourage collaborative investigations of geometrical properties.

We have discussed designing tasks and are ready to discuss implementing tasks. Some tasks are designed by teachers. Many are not. Whether or not the designer and implementer are one and the same, implementation of a task does not necessarily follow the intention of the designer. That is, how a task is meant to be implemented is not necessarily what happens in actuality. This is discussed in the next section.

## 4.3 IMPLEMENTING MATHEMATICAL TASKS

Donna and Nina teach in two different kindergartens located in the same neighborhood. Both teachers implement the same "pull out all the triangles" task using the same set of figures in their kindergartens. If we go and observe both teachers as they implement this task, can we expect to see more or less the same activity? How might you implement this task with kindergarten children?

According to Styliandes and Styliandes (2008), the implementation of tasks by teachers is mediated by the teachers' beliefs, knowledge, and previous teaching experiences. For example, teachers with weak content knowledge may not always understand the educational goals of tasks and appreciate the mathematical appropriateness of students' solutions. On the other hand, strong mathematical knowledge may not be enough to ensure the fidelity of a high-level task. In their study of a secondary mathematics teacher Styliandes and Styliandes found that even though the teacher had the necessary mathematics knowledge to implement a high-level task, the custom or norms of the classroom hindered implementation of the task at the high level it was designed for. Similarly, Watson and Sullivan (2008) pointed out that most teachers need to adapt their teaching customs to the school's cultural practices. Thus a task designed to foster collaborative investigation may not be implemented with fidelity in a school where 'chalk-and-talk' teachers' methods are the norm. Eisenmann and Even (2009) also found that implementation of a task may be related to classroom culture. In their investigation of one teacher who taught seventh grade algebra in two different schools, they found that even when the same task was implemented by the same teacher, in one class the task was enacted as a global/meta-level activity but in the second class it was not. Several possible reasons were offered for this difference. First, in one school, the teacher was more autonomous and was allowed the freedom to spend more time on tasks of her choice. This was not so in the second school which had a more rigorous timetable. Also, in the first school children were used to working on tasks in small groups and thus used their time more productively, enabling them to reach the meta-level of activity inherent in the task. Finally, the culture of the first school was to continuously strive for excellence so that teachers were encouraged to challenge students and used the tasks given to them to reach higher levels of thinking.

In a similar manner, the culture of the kindergarten class and the teacher's knowledge and beliefs might influence the way in which a task is implemented. For example, is geometry seen as a "by-the-way" subject to be learned while taking part in "natural" play or is it also part of teacher-directed tasks? Is whole class time spent on story telling or is it also spent on discussing mathematics and geometry? Is mathematics and geometry something "to do" or is it also something "to be discussed"? On the one hand, the "pull out all the triangles" task may be seen as something "to do". The child must look through the bin and physically pull out all of the triangles. On the other hand, this same activity may take place in a group setting where the teacher passes around the bin and each child in his turn must pull out a triangle and explain to the group and to the teacher why the cutout is indeed a triangle. In this way, geometry becomes also something to be discussed.

Is it possible to insure the fidelity of implementing a task by providing the teachers with explicit instructions? Bieda (2009) observed middle school teachers and their students who participated in the Connected Mathematics Project. This curriculum incorporated proof-related tasks which the teachers could then implement. Yet, even though most teachers were observed implementing the tasks as written, not all students' generalizations were followed up with justifications. When justifications were not forthcoming, many teachers did not provide necessary feedback, missing the opportunities provided by the situation to engage students in proving activities. In other words, even when the teachers were provided with tasks that encourage proving, implementing the proof part of the activity did not necessarily follow. Many kindergartens and preschools are supplied with mathematical and geometrical materials as well as instructions on how these materials may be used as tools to promote geometrical learning. And yet, are they used in the intended manner?

It is common knowledge among professional development providers and researchers that supplying teachers with materials and instructions is not enough. As such, in several of our professional development sessions we spend time discussing, designing, and even enacting different tasks that may be implemented with various materials. Together with kindergarten teachers we evaluate these tasks, their mathematical aims, and possible outcomes.

The following is an excerpt taken from a professional development session held with prekindergarten and kindergarten teachers which explicitly discussed the design and implementation of geometrical tasks in kindergarten.

Instructor: What types of things will you put in the geometry corner? What types of materials?

Joy: Most of the (commercial) games that we buy have many different shapes. There isn't one game which focuses on just one shape. So, if you want to just work on triangles …

Instructor: Maybe we have to construct and design our own tasks.

Joy: Like finding lots of different triangles in different situations and sorting the triangles. We could use acute triangles and obtuse triangles.

Instructor: What would be the advantages and disadvantages in sorting only triangles? Maybe the task should include lots of different figures and then the children can make two groups, one of triangles and one of non-triangles. For example, we can draw shapes that look like triangles but are missing one attribute, like an open triangle, like we discussed together in the lesson. If we use only triangles, then we won't know if the children can differentiate between examples and nonexamples of triangles.

Hanna: And when the child says that it's like a triangle, you can ask him, but why isn't it a triangle?

Joy: And we can add a sign to the activity board, which says in big letters, triangle, and draw next to it a triangle, and underneath it we can write the definition of a triangle and draw the different parts. (See Figure 2.)

Instructor: We should also build the task so that each critical attribute gets attention. Emphasize the three as opposed to not three. Closed and not open.

Hanna: So, let's think of which examples and nonexamples we want to use and then we can all have a set to use in the class.

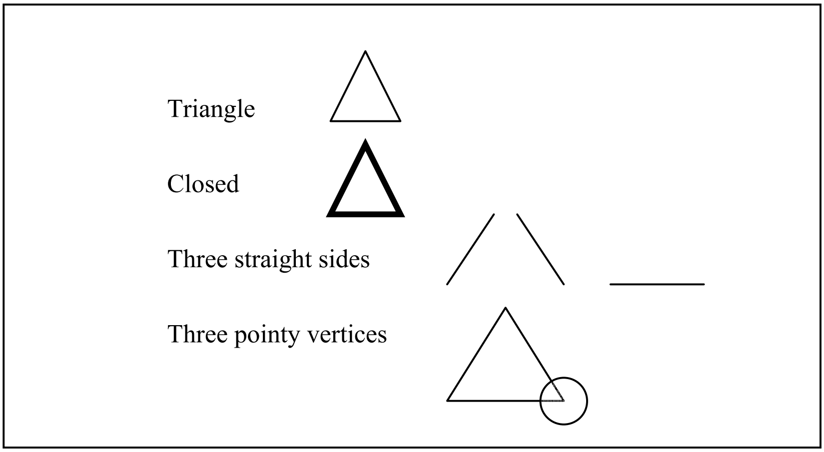



*Figure 2. Activity board sign with icons representing the critical attributes of a triangle.*

The teachers begin by thinking of commercial materials they have seen in the past which may be used when engaging the children in geometrical tasks. These materials and games often include various prototypical shapes, such as equilateral triangles, squares, and circles. They are less likely to include non-intuitive examples and nonexamples of a shape. The teachers agree that if they want to focus on triangles, they should consider designing their own tasks and materials. The teachers, along with the instructor then discuss the affordances and constraints of a task which makes use of only examples as opposed to a task which takes into consideration also nonexamples. Joy considers how providing a guiding source may be helpful to the children. Hanna considers how the children may react and the necessity for children to understand that if a figure is "like" a triangle it is not the same as saying it "is" a triangle. She is also eager to build a set of figures together in the course which is exactly what they did during the rest of the lesson. Later in the week, when the instructor went to visit the different kindergartens, all of the teachers except for Joy displayed the different examples and nonexampels that children were sorting. Only Joy chose to place only examples of triangles on her activity board, demonstrating that even when a task is designed and discussed, implementation may differ from teacher to teacher. It could be that Joy felt that the children in her kindergarten, some who were known to be learning disabled, would not be able to handle examples and nonexamples at the same time.

The above scenario illustrates how professional development may be used to discuss some of the elements of task design and implementation. This is especially helpful when considering that many preschool teachers may have had little experience implementing mathematical tasks, including geometrical tasks, in the past. Professional development for preschool teachers is the focus of Part Three in this book.

After having discussed various theoretical issues related to the design and implementation of mathematical, as well as geometrical tasks, we are now ready to take a critical look at some specific geometrical tasks.

## 4.4 GEOMETRICAL TASKS FOR YOUNG CHILDREN: CURRICULA GUIDELINES

In this section we describe a few geometrical tasks that are recommended by various mathematics curricula. In general, when curricula mention tasks, they do so in order to illustrate possible ways in which the child may engage in activities which may promote learning certain concepts. In these curricula, in general, one will not find a rigorous analysis of the tasks, their design, implementation, affordances, and constraints, as we detailed above. We bring you examples of tasks mentioned in different curricula in order to provide general ideas of tasks mentioned in various countries. As you review these tasks you may ask yourselves what mathematical activity is implicit in the task. What are the affordances and constraints of the task? What messages may the children receive regarding geometry when engaging in this task?

Many curricula guidelines offer learning goals, what the final results should be, what children are supposed to know by a certain age. Yet, they do not offer specific examples of tasks. In England, the Statutory Framework for the Early Years Foundation Stage (EYFS) (2008) states that by the end of this stage (5 years of age) a child should be able to use “mathematical language to describe solid (3D) objects and flat (2D) shapes” (p. 47). In the accompanying non-statutory practice guidance, although specific tasks are not delineated, it is suggested that the practitioner provide an area where children can explore the properties of objects, plan opportunities for children to describe and compare shapes, and make books about shapes found in the environment. Taken together, these suggestions give the overall feeling that it is important to provide opportunities for learning geometrical concepts. However, providing opportunities does not necessarily translate into designing and implementing tasks. It would seem that it is up to the preschool teacher to design tasks or to choose tasks from curriculum materials advertised in catalogues and trade magazines.

Some national guidelines offer samples of tasks, along with aims and guidelines, which may be implemented at different ages. The Israel National Mathematics Preschool Curriculum (INMPC, 2008) states that by the end of kindergarten children should be able to identify and name two- and three-dimensional figures as well as describe their attributes, such as the number of sides and vertices of a polygon. Alongside these aims, the curriculum suggests tasks that may be implemented with children. For example, they suggest that geometrical shapes may be used in order to construct pictures. Another suggestion is for the teacher to present the child with a picture constructed from different geometrical figures. The child is then requested to color all the rectangles red, the triangles, blue, and the circles green. What might be the affordances and constraints of these tasks? Certainly, both of these tasks are constrained by the different shapes the teacher chooses to use in the task. What does it afford? Using shapes to construct pictures allows children to manipulate and rotate different figures, perhaps enlarging their concept image of geometrical shapes to include those with different orientations. Identifying shapes "hidden" in a picture may serve to refine children's ability to see differences and similarities between different shapes.

In the United States the Principles and Standards (NCTM, 2000) state that instructional programs from prekindergarten through grade 2 should enable all students to "recognize, name, build, draw, compare, and sort two- and three-dimensional shapes, describe attributes and parts of two- and three-dimensional shapes, and investigate and predict the results of putting together and taking apart two- and three-dimensional shapes" (p. 97). Suggested examples of tasks that encourage these aims are also provided. One such task involves constructing triangles on a geoboard. Using one band per triangle, the child is requested to make many different sizes and shapes of triangles and explain to a friend the ways in which these triangles are different and how they are alike. The aims of this task are to explore the concept of triangle focusing on the properties of triangles while also paying attention to congruence. That the curriculum specifically adds that the child should explain his or her productions to a friend, tells us that this task is to be implemented in a group or at least with two children. It also lets us know that the mathematical activity involved in this task is more than constructing shapes. It includes explanations and communication of mathematical ideas such as congruence. This task may also promote the belief that geometry is something to do as well as something to discuss.

In Australia, the New South Wales Mathematics K-6 Syllabus (2006) details over 10 explicit geometrical knowledge and skills aims for kindergarten children. These include sorting two- and three-dimensional shapes and objects according to features, identifying and naming circles, squares, triangles, and rectangles in different orientations, and recognizing and using informal names for three-dimensional objects. In the accompanying sample units of works there are explicit examples of tasks. One such example recommends a task to be implemented by pairs of children. Each student in the pair is given an identical set of two-dimensional shapes. Student A creates a flat design using the shapes and hides it from Student B. Student A must describe his design to Student B who then has to recreate it with his own shapes. In the manner in which this task is described, as well as the manner in which many tasks are described in different curricula, much is left up to the teacher to decide. What shapes will the teacher use? Will she supervise the task? Is the emphasis on doing or describing? What will be the balance between security and insecurity?

## 4.5 SUMMARY

In this chapter we reviewed some elements of mathematical task design as well as geometrical tasks mentioned in curricula guidelines. Elements of this review are referred to in the upcoming chapter which discusses in depth two geometrical tasks which may be implemented with preschool children.

CHAPTER 5

# IMPLEMENTING GEOMETRICAL TASKS – SOME POSSIBLE SCENARIOS

Four 3-year old children are sitting around the table with the teacher-researcher. Each child, in turn, receives a card with a drawing of a shape on the card. The task for the child is to tell everyone if the shape on the card is or is not a triangle and why.

Teacher to Pam: Here is your card. Is it (pointing to the drawing on the card) a triangle?

Pam: Yes.

Nadine: Where is my triangle?

Teacher to Nadine: Soon, you'll also get a shape. (Turning back to Pam the teacher continues.) It is a triangle. Correct. Why is it a triangle?

Pam: Because it has three vertices.

Teacher: Can you show them to me?

Pam: (Pam points and counts each vertex.) One, two, three.

Teacher: Fine. And what else does a triangle have?

Nadine: Sides!

On the one hand, the task above was designed to be implemented in a group so that the child has a chance to show his or her friends what was found and to explain this finding. On the other hand, Nadine is an impatient child. It is hard for her to wait for her turn. Or, maybe she is an enthusiastic child. She also wants to join in and the teacher at this point is only paying attention to Pam. Would you say that the above task was designed to be implemented with an individual or with a group? It is hard to tell. Would you implement this task with an individual or with a group? Perhaps it depends on the age of the children or on the specific circumstances of that day's routine. Implementation of the same task with an individual has affordances and constraints different from implementation with a group. In the previous chapter we discussed, in theoretical terms, the elements of a task and what needs to be considered in its design. We focused on the task itself. In this chapter we focus on two geometrical tasks and their affordances and constraints. We present these tasks as if they are to be implemented with an individual, even though one can easily imagine, as the above scenario illustrates, the same tasks being implemented with a group. By approaching the task as if it is being implemented with an individual child we are able to focus on the nuances of the task, how the task may be used to investigate as well as promote a child's geometrical knowledge.

At the heart of each task are the examples and nonexamples presented to the children for various geometrical figures. In Part one of this book, we discussed in depth how examples and nonexample may contribute to concept formation. Here, we continue this discussion on a practical level, pointing out for each task how the examples and nonexmples, as well as the sequence of their presentation, may impact on concept formation.

For each task, we begin by presenting the general mode of implementation. We then present possible scenarios depicting how the task may play out, including some results from our various studies where the tasks were implemented in preschools. We summarize each task by pointing out the affordances and constraints of the task.

## 5.1 TASK ONE: ONE SHAPE AT A TIME

*Implementing the One Shape at a Time task*

Consider the following figure (see Figure 1).

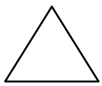


*Figure 1. A prototypical triangle.*

Is it a triangle? Why? Now reflect on your answers. Did you answer the first question immediately or did you hesitate somewhat? Did you explain (to yourself) why the figure is a triangle? Did it seem silly to offer an explanation when the figure is so obviously a triangle?

Consider the shape in Figure 2.

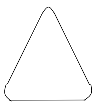


*Figure 2. Rounded corner "triangle".*

Is it a triangle? Why? Now reflect on your answers. Did you answer the first question immediately or did you hesitate somewhat? Were you able to explain why this figure is not a triangle?

It is quite possible that your answers to both the geometry questions as well as the reflection questions may be different if you are a mathematics education researcher, preschool teacher, or the parent of a preschooler. Now consider how a preschooler may answer the geometry questions. Would he be able to correctly identify the first figure as a triangle? How would he explain his identification?

Would he correctly identify the second figure as a non-triangle? How would he explain his identification?

Task one, One Shape at a Time, is a task which focuses on two central elements of geometrical knowledge: identification and reasoning about geometrical figures. Cards are made up with a drawing of one figure per card. If we are currently concerned with triangles, then the figures will consist of various examples and nonexamples of triangles. If we are currently concerned with hexagons, then the figures will consist of various examples and nonexamples of hexagons. The child is then presented with one card at a time and asked: Is this a triangle? Why? (Or, if we are currently concerned with hexagons, then the child is asked: Is this a hexagon? Why?) The same questions are repeated for each card. Presented in this manner, the task allows the child to focus on one shape at a time, both in terms of his focus on the one drawn figure on the card and in terms of his focus on the one figure, triangle (or hexagon), in question. It also allows the teacher or researcher to listen carefully to the child's response to each figure, noticing the immediacy or possible hesitancy in the child's response, and proceed appropriately.

*Possible scenarios when implementing the One Shape at a Time task*

Let us consider the following scenarios. Nancy is the teacher in a preschool for 4-5 year olds. She has previously presented to her children the following working definition of a triangle: A triangle is a closed figure with three pointy vertices and three straight sides. During whole class time, the children have learned to identify vertices, straight and curved lines, open and closed figures but they have not yet had time to practice using these concepts when identifying various triangles.

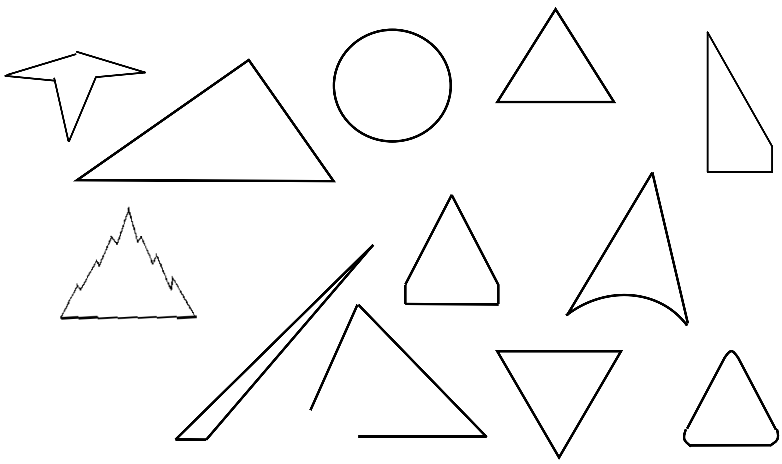


*Figure 3. An assortment of triangles and non-triangles chosen by Nancy*

She wants to engage her young students in this task in order to widen their example space of triangles and encourage them to use critical attribute reasoning when identifying triangles. Towards this end, she has created a wide variety of cards with examples and nonexamples of triangles, one figure to a card. A sample of these figures is shown in Figure 3.

Nancy has decided that at first, she will begin with the prototypical triangle (see Figure 4) and then, depending on the child's response, she will decide what card to present next. What follows are three possible responses to the prototypical triangle from three different children.

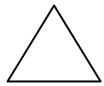


*Figure 4. First triangle presented by Nancy.*

C1: Yes, it's a triangle because it looks like a triangle.
C2: Yes, it's a triangle because it has vertices and straight sides.
C3: Yes, it's a triangle because it has three vertices.

How to react to each child? What figure should Nancy present next to each child? C1 has correctly identified the prototypical triangle but has given a visual explanation. It seems that C1 is operating at the first van Hiele level of reasoning. The teacher may take the opportunity with this easily recognizable triangle to review the critical attributes of this and all triangles. And then, what figure should be presented next? Should it be an example or a nonexample? Let us say that we continue with an example. Which of the following examples shown in Figure 5 might be presented next in order to encourage C1 to move beyond visual reasoning?

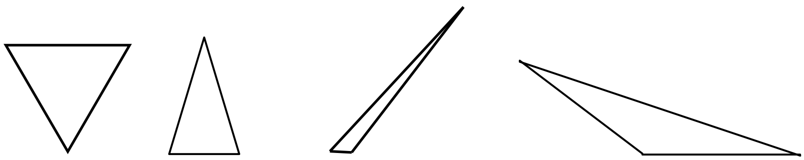


*Figure 5. Upside down prototypical triangle, isosceles triangle, scalene triangle, obtuse triangle.*

For sure, there is no one correct answer. Each choice will probably provide the teacher with additional knowledge regarding C1. Does C1's example space include triangles of different orientations? Of different angle measures? Perhaps C1's concept image includes a variety of triangles but the need to use attribute reasoning when identifying each triangle is not always felt. While there is no one correct answer, our choice would be the scalene triangle. This triangle is visually far removed from the prototypical triangle. If the child does identify the scalene triangle correctly, it is unlikely that an accompanying explanation would be "it looks like a triangle". It is more likely that because this triangle is far removed from the prototypical triangle, the child will turn towards the concept definition, checking for critical attributes that must be present. In other words, the child who is knowledgeable of critical attributes may well not feel the necessity to use them when identifying prototypical shapes but may be inclined to do so when identifying non-prototypical shapes. Studies have shown that the van Hiele levels are not necessarily discrete and that a child may operate at two different levels depending on the task, the context, and the examples and nonexamples presented (Burger & Shaughnessy, 1986; Clements & Battista, 2001). Thus, we may use this task and order the presentation of examples so as to encourage the child to use reasoning more appropriate for the second van Hiele level rather than the first. Did you choose a different example? Why? Perhaps for even younger children, you may feel that at first it is important to work on orientation. Perhaps you feel that a nonexample should be presented next.

What nonexample may be presented to C1 after the response given? Consider the following two nonexamples in Figure 6:

*Figure 6: Square, rounded "triangle".*

Which would you present next? Why? The square is intuitively recognized as a non-triangle (Tsamir, Tirosh, and Levenson, 2008) and thus may not lead the child to use an explanation incorporating critical attributes, even if C1 is aware of the critical attributes. In other words, presenting the square may not discern for the teacher if C1 is capable of reasoning with critical attributes. On the other hand, the rounded triangle looks like a triangle but is not. Therefore, if the child does not discern the critical attribute of pointed vertices and relies on visualizing the whole shape, this figure will be incorrectly identified as a triangle. However, if C1 is aware of the critical attributes of a triangle, the need for referring to them when identifying the prototypical triangle may not have arisen whereas at this point, when a figure which so closely resembles a triangle is presented, the need to refer to the critical attributes may be stronger.

Now consider the response given by C2: "Yes, it's a triangle because it has vertices and straight sides." Unlike C1, this explanation is not based on whole shape recognition. Instead, C2 takes note of the vertices and straight lines. Does this explanation satisfy you? Would you accept this explanation as sufficient? How best to proceed? Will a different example help C2 take note of the number of vertices or the number of sides? We are not so sure. Perhaps. Instead, we would choose to present C2 with a nonexample next. Consider the following nonexamples in Figure 7:

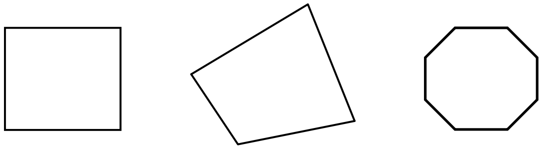


*Figure 7. Square, non-square quadrilateral, octagon.*

Which one would you present next to C2? Why? As we noted in the previous scenario, there is no one correct answer. In Book 1, Chapter 2, we discussed at length how to present different nonexamples in order to allow the child to focus on one critical attribute at a time. In this scenario, we are not sure if C2 is aware that threeness is critical but has just not mentioned it or if he is truly unaware of this critical attribute. Of course, the teacher may directly ask C2 to count the number of vertices. This may raise to the fore that the triangle has three vertices but will C2 conclude from counting the vertices that having exactly three is a critical attribute? The beauty of this task is that it allows the teacher to use examples and nonexample to create cognitive dissonance in the child. C2 took note of the vertices and sides. By presenting him next with a square, which also has vertices but is intuitively recognized as a non-triangle, we raise the issue of the number of vertices. A square also has vertices and sides but it has four, and not three, vertices and sides. Of course, the square may be problematic because being an intuitive nonexample, the child may not feel the need to refer to attributes at all. Perhaps a non-square quadrilateral would be better. What about the octagon? The octagon has eight vertices. Perhaps this would be good a nonexample to follow the prototypical triangle? On the other hand, even though children may be capable of counting till eight, they may simply view eight as a lot or as many (Sarnecka & Gelman, 2004). In other words, the octagon may have too many vertices and therefore it may not create enough dissonance for the child to specifically note that the triangle has exactly three vertices and three sides.

Have you considered the age of C2? Not all three-year old children may comprehend the principle of cardinality. For example, Baroody and Wilkins (1999) cited an episode where a three-year old successfully counted four stars on a card but when asked how many stars were on the card, she shrugged her shoulders and counted again from the beginning. Perhaps for such a young child, it would be helpful to count the number of vertices and the number of sides noting each time the number three. It may then be helpful to present the three-year old with additional examples of triangles noting over and over the number three and only then switch to polygons with four and possible five vertices.

Consider now C3's response to the prototypical triangle: "Yes, it's a triangle because it has three vertices." Does his explanation satisfy you? How might you continue? Would you show C3 another example or a nonexample? At this point, we know that C3 is aware that a triangle has three vertices. Does C3 know that the triangle must also have three straight sides? Most young children are not aware of the relationship between sides and vertices, that three sides necessarily follow three vertices. Does C3 know that the triangle must have three and not four vertices? We suggest presenting one of the following nonexamples shown in Figure 8. Each of these nonexamples refutes a different critical attribute and may raise C3's awareness of other critical attributes.

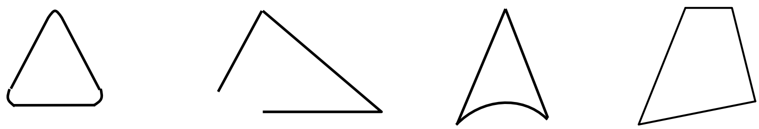


*Figure 8. rounded "triangle", open "triangle", "clown hat", quadrilateral.*

The "rounded triangle" is missing vertices. Perhaps you feel that this nonexample is extraneous considering that C3 specifically mentioned three vertices. On the other hand, if we are not sure of C3's conception of vertices, this nonexample may serve us well. Perhaps C3 is not bothered by the roundness of the corners. The other nonexamples raise different issues. The "open triangle" may remind C3 that a triangle must be a closed figure and the "clown hat" may remind C3 that sides must be straight. The quadrilateral has four and not three vertices and sides. On the other hand, you may want to know if C3 always refers to the vertices first or if it was just a coincidence. In that case, it may be interesting to follow the first example with more examples and investigate whether C3 offers the same explanation for each triangle, perhaps indicating that, for C3, the attribute of vertices is more dominant than other attributes.

In the above section, we presented three different scenarios for how the task may play out. There are as many possible scenarios as there are children. Often the same task may be used to both promote as well as evaluate children's knowledge. In the next section, we present results of a study which implemented this task at the end of the school year in order to assess kindergarten children's knowledge of pentagons. As we shall see, implementing this task on a one-to-one basis with children also allowed us to investigate the relationship between children's geometric knowledge and their monitoring behaviors.

*Results of a study: Is this a pentagon? Why?*

The children in this study had learned in preschools where the teachers had participated in our two-year professional development program Starting Right: Mathematics in Preschools. We discuss this program in more detail in Part Three of this book. However, at this point we note that the teachers spent a great deal of time during the year on extending children's concept image of triangles. In addition, they introduced the children to a working definition of triangles such as that proposed in Part One of this book. Thus children were aware that triangles must be closed and have three and only three straight lines and pointy vertices. Only after the teachers were satisfied that children were able to use attribute reasoning when identifying triangles, did they introduce the children to pentagons and later still, hexagons. Throughout the program, each teacher was personally accompanied by a member of the professional development staff who visited the preschool on a weekly basis, sitting with children and guiding the teacher in her endeavor to create a mathematically enriched environment for her children. The aim of the program was for all children to exhibit competency in all mathematical areas taught (such as geometry, numbers, and operations).

The study we report on here took place towards the end of the school year with 182 5-6 year old children in the year prior to their entering first grade. Each child was presented with six different shapes, one shape at a time, and asked to identify each of the shapes as a pentagon or a non-pentagon. The child was then requested to justify the identification. At times, children's initial identifications remained unchanged and at times children's final identifications differed from that of their initial identifications. What follows is a review of typical responses to one pentagon shape and to one non-pentagon shape (see Figure 9). Responses are categorized according to correctness of identification as well as the type of reasoning which accompanied children's identification.

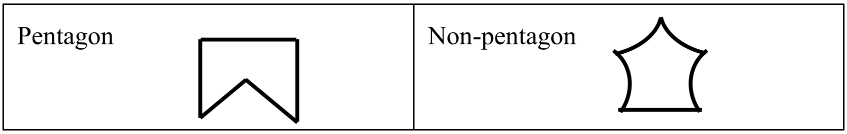



*Figure 9: Two shapes presented to children for the pentagon task.*

*Correct initial and final identifications with critical attribute reasoning*. Regarding the pentagon, children who identified this shape correctly often justified their identification by referring to critical attributes of the pentagon.

C1: It has five vertices, it's a closed shape, and it has five straight lines.

Regarding the non-pentagon, some children who correctly identified this shape as a non-pentagon referred in their justifications to "curved" or "rounded" lines. One child justified his correct identification by saying, "It's not (a pentagon) because it has two rounded sides … actually is has four rounded sides … it doesn't matter." This child assessed his justification "on line". At first he noticed two rounded sides. Then he took a closer look and noticed four rounded lines. However, he realized immediately, that in fact it does not matter how many rounded sides the shape has, because even one is sufficient to nullify the shape as a pentagon. This child exhibited monitoring, not of his solution (which was correct) but of his justification. As he was justifying his conjecture, he monitored the correctness and perhaps quality of his justification.

Regarding both shapes, some children first counted the vertices or sides and only then responded to the question of identification. Such children thought about how to go about identifying the shape, acted on their plan, identified the shape and then justified their identification.

*Incorrect initial identification but correct final identification with critical attribute reasoning*. Children who corrected their initial incorrect identifications, typically referred to the critical attributes of a pentagon in their justifications. Regarding the pentagon (see Figure 9):

> C2: It's not a pentagon. Let's check. (The child counts the vertices.) It is a pentagon because it has five sides and five vertices and it's closed.

> C3: It's not a pentagon. The line here points to here (referring to the concaveness of the pentagon). (The child counts the vertices.) It is a pentagon.

C2 immediately went to check his conjecture, even before the researcher had a chance to ask him why he claimed the shape was not a pentagon. In other words, he initiated the monitoring (when he declared "let's check" and counted the vertices) which in turn led to a correct identification based on a correct justification. C3 initially used a justification based on a non-critical attribute (the direction of the line). This justification was followed by monitoring (counting the vertices) which in turn led to a correct identification and an explanation based on properties and critical attributes.

Regarding the non-pentagon (see Figure 9), one child claimed at first that this shape was a pentagon. When asked why he thought it was a pentagon, he proceeded to count the points and said, "Yes … uh … no. It has five vertices but it's not straight." One might think that the child changed his mind because the interviewer challenged him. However, in this kindergarten, asking children to explain their reasoning, regardless of whether an answer was correct or incorrect, was a norm previously established. Thus, it is more likely that the act of justifying the conjecture led to self-initiated monitoring.

*Incorrect initial and final identification with critical attribute reasoning*. At times, children gave incorrect identifications along with critical attribute reasoning. For example, regarding the pentagon (See Figure 9):

> C4: It's not a pentagon. It doesn't have five sides. (There was no indication that the child had counted the sides.)

It seems that C4 gave a verbal justification without carrying out any action. Although he gave a justification befitting his (incorrect) identification, the request for justification did not lead this child to monitor his response. He did not look back and was not aware of his mistake.

*Unchanging identifications (correct and incorrect) with visual reasoning*. Not all children justified their identifications using the critical attributes of a pentagon. Regarding the pentagon:

> C5: It's a pentagon because it looks like a pentagon.

> C6: It's not a pentagon because it looks like a tooth.

C7: It's not a pentagon because it doesn't have the shape of a pentagon.

Regarding the non-pentagon:

C8: It's not a pentagon because it looks like a circus (tent).

C9: It's not a pentagon because it's not in the shape of a pentagon.

The above children used visual reasoning in their justifications. Both C6 and C8 embodied the rather abstract concept of a pentagon into a more familiar physical entity. C5, C7, and C9 have a concept image of a pentagon which does not fit the shape on the card. These justifications accompanied both correct and incorrect identifications and were not accompanied by monitoring.

Some children gave justifications that were a mix of perceptual reasoning along with reasoning based on attributes. Regarding the non-pentagon:

C10: It's not a pentagon because it has five vertices but it doesn't look like a pentagon.

C10 seems to be in transition. Previously, he had correctly identified the pentagon noting only its five vertices. His justification regarding the non-pentagon takes note of the five points (they are not vertices as they do not connect straight lines), but disregards them because the shape "doesn't look like a pentagon." In other words, he realizes that the attribute of "pointy vertices" is worthy of notice but he may not have the knowledge or words to describe that the sides need to be straight lines. Instead, his final justification relies on his visual perception. In a sense, C10 exhibits monitoring. He clearly has a strategy by which he checks if a shape is a pentagon (counting vertices) but "on line" rejects that reason in favor of relying on his mental image of what a pentagon should look like.

### *Summarizing the One Shape at a Time task*

In summarizing this task we refer to Watson (2004) who discussed the affordances and constraints in mathematical tasks. The One-shape-at-a-time task affords the teacher and researcher flexibility in determining the course of interaction with the child and the ability to tailor the task to meet the needs of different children at different points in their development. For the learner, the One-shape-at-a-time task affords time to focus on one figure and examine it closely, time to review the working definition and check the critical attributes of the shape being discussed against the figure being presented. It also provides the learner with the opportunity to investigate a variety of examples and nonexamples and to use the working definition of the shape being discussed when explaining one's identification.

What are the constraints of this task? The examples and nonexamples used obviously constrain what may be learned from this task, what children may learn about triangles as well as what we may learn about children's knowledge of the figure at hand. In addition, there is a certain repetitiveness to this task which may possibly constrain meaningful learning. Once a child has learned to explain that a figure is a triangle because it has 3 vertices, 3 sides, and is closed, he may use this explanation repeatedly without thinking of what he is saying. When one task becomes routine it may be wise to move on to a different task. In the following section we present a different geometrical task which may be implemented with young children.

## 5.2 TASK TWO: WHAT DO WE HAVE HERE?

### *Implementing the What Do We Have Here? task*

Whereas the previous task setting presented the child with one shape at a time, this task presents the child with an assortment of figures all at once. We place before you the following shapes in Figure 10. Do you see a triangle? Yes? Point to it. Do you see another triangle? Yes? Point to it. Do you see another triangle? No? Do you see a quadrilateral? Yes? Point to it. No? Do you see a pentagon?

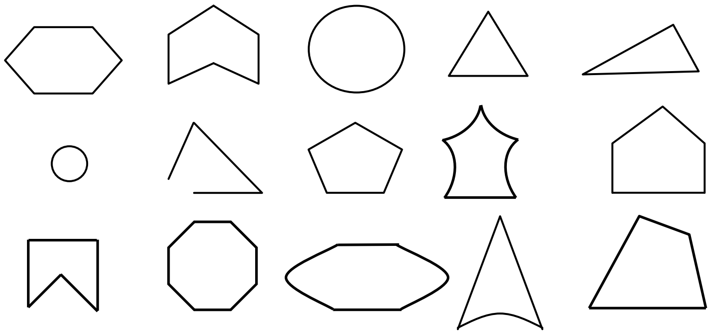


*Figure 10: An assortment of shapes.*

The *What-Do-We-Have-Here?* task challenges children to seek out a single shape among many figures presented at once. The teacher or researcher places a variety of figures in front of the child, chooses a shape to be identified, for example a triangle, and asks, “Is there a triangle here?” If the child answers yes, then the child is asked to point to the appropriate card without removing it or changing its position. The teacher or researcher then asks, “Is there another triangle here?” And again, the child is asked to point to it. This continues until the child indicates that there are no more triangles. The teacher may then move on to a different shape, and the same set of questions with the same procedure is repeated.

There are several challenges for the child in this task. First, the child must deal with a variety of figures at once instead of focusing on one figure at a time. Depending on the age of the child, this can be very difficult. Adding to this difficulty is the teacher’s request to point to the shape and not remove it. In other words, the child cannot remove the figure and physically rotate it. Instead, the child must deal with a static picture. In addition, the identified figure remains among the assortment, keeping the amount of figures static. Of course, an easier variation of this task would be to request that the child remove the identified shape from the assortment, thus decreasing the amount of figures to be reckoned with each time. On the other hand, if the child incorrectly identifies a shape and removes it, then that shape will be missing when it is sought out later on. These variables may be adjusted by the teacher or researcher according to each situation. An additional challenge is requesting the child to seek, for example a triangle, after he has already identified all the triangles. Most children learn, rather early on, that when the teacher requests something to be done, it can be done. Thus, asking the child to point to a triangle when he has already pointed to all of the triangles may cause him to hesitate. Yet, not all problems, including mathematical problems, have a solution. Therefore, it is important to establish a norm, as early as possible, whereby children may confidently claim that a problem has no solution. This task provides such an opportunity by continuing to ask for triangles, even after all the triangles have been identified. The teacher may also deliberately request a square, for example, when to begin with, no squares were placed among the assortment of figures.

This brings us back to the variety of figures presented to the child at one time. How to choose the assortment? First, one needs to think about which geometrical figures will be the focus of this investigation. Will the teacher request the child to point out three different figures, say triangles, quadrilaterals, and pentagons? Or are the children at the stage when they can be asked to identify triangles, quadrilaterals, pentagons, hexagons, and circles? Perhaps for very young children, it is enough to present a variety of figures and ask the child to point out only the triangles. It also needs to be decided how many examples of each figure will be among the assortment of figures presented to the child. For example, we most likely would include at least one prototypical or intuitively recognized example for each figure and one less so. Regarding the nonexamples, we first consider that an example of a triangle is a nonexample of a pentagon. Thus, the examples we have chosen for the various figures, may also serve as nonexamples for other figures. We then might include other nonexamples which refute different critical attributes. On the other hand, if we include an "open triangle" then we may not feel the need to include an "open pentagon". Finally, the total amount of figures displayed at once must be considered, especially in light of the young age of the children.

*Possible scenarios when implementing the What Do We Have Here? task*

Rachel presented the following 12 shapes (see Figure 11) to her preschool children. Notice how the figures are all examples of either a triangle, square, or circle. Consider for a moment what possible scenario might call for such an arrangement. What might be the ages of the preschool children? At what stage of their geometrical development might they be?

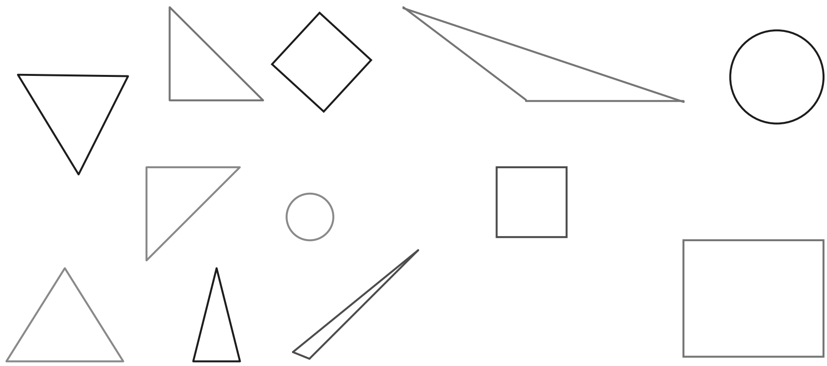


*Figure 11: Rachel shows these figures to her preschool children. The figures are in many different colors.*

Rachel is a preschool teacher for 3-year old children who has chosen to focus her attention on recognizing triangles, squares, and circles as well as naming these shapes. During circle time, she has presented the children with one shape at a time, focusing on expanding the children's concept image to include shapes of various sizes, colors, and orientations. She would like at present to introduce her children to the critical attributes of a triangle but wishes first to assess how the children cope with many shapes at once. In consideration of the very young age of the children, she implements this task with a variation. Instead of having the child point to a figure, she requests the child to pick up the requested figure and hand it to the teacher.

Rachel: Lily, can you hand me a triangle?

Lily hands the teacher the isosceles triangle.

Rachel: Can you hand me another triangle?

Lily hands the teacher a right triangle.

Rachel: Can you hand me another triangle?

Lily hands her the upside down triangle.

Rachel: Can you hand me another triangle?

Lily hands her a square.

At this point, the teacher is confused. It seemed as though Lily was doing really well. Why did she hand the teacher a square? Perhaps she does not recognize the scalene or obtuse triangles as triangles, but surely the prototypical triangle should be identified as a triangle. Perhaps, due to the multitude of figures, Lily has just missed the prototypical triangle. But if that's the case, then why did Lily hand the teacher a square? Does she not know the name square? And if she does not know the name for a square does that mean that Lily thinks that this may also be a triangle? Due to the multitude of questions, Rachel decides to not continue but to inquire of Lily why she handed her a square.

Rachel: Lily, is this (holding up the square she received from Lily) a triangle?

Lily: No, it's a square.

Rachel: Ok. Can you now give me another triangle?

Lily hands the teacher the scalene triangle.

Rachel continues on with the task and Lily correctly completes the entire task. What can we say about Lily? She correctly identified all of the figures, including the non-intuitive triangles and square. Why did Lily seemingly make the mistake described above? We are not sure. We are not even sure that the three-year old would be able to tell her teacher why she handed her a square even as she knew it was not the requested triangle. However, this scenario demonstrates how we cannot always judge a child's knowledge based on one response and that, especially with young children who have not yet developed the habit of being "tested", intervening in the middle of a task, even an assessment task, may be a necessity. This scenario also demonstrated how a task may be varied and adapted to fit the needs of the teacher and the children. In this case, the figures were all examples of previously learned shapes. Thus, the nonexamples of triangles were considered intuitively recognized as nonexamples. In addition, having the child hand the requested figure to the teacher, lessened the cognitive burden, disallowing the child to inappropriately choose a figure twice. On the other hand, the teacher continued to ask for triangles even when only squares and circles were left, underscoring the importance of teaching children at a very young age, that not all problems may have a solution.

Karen teaches kindergarten children who will be entering first grade in the upcoming school year. During the year she has taught her children about circles, triangles, quadrilaterals, pentagons, and hexagons as well as the critical attributes of the above mentioned polygons. Towards the end of the year, she is interested in assessing her children's knowledge regarding polygons. She has collected the following polygons and presents them to Tracy (see Figure 12).

Karen begins her query with triangles, and then moves on to quadrilaterals, pentagons, and hexagons. Without hesitation, Tracy points to a different triangle each time Karen inquires if there is yet another triangle and replies that there are no more triangles after she has identified all of the triangles. When it comes to identifying quadrilaterals and pentagons it takes her a bit more time as she sometimes pauses to count vertices. Still, she identifies all of the quadrilaterals and pentagons. Karen then moves on to the hexagons.

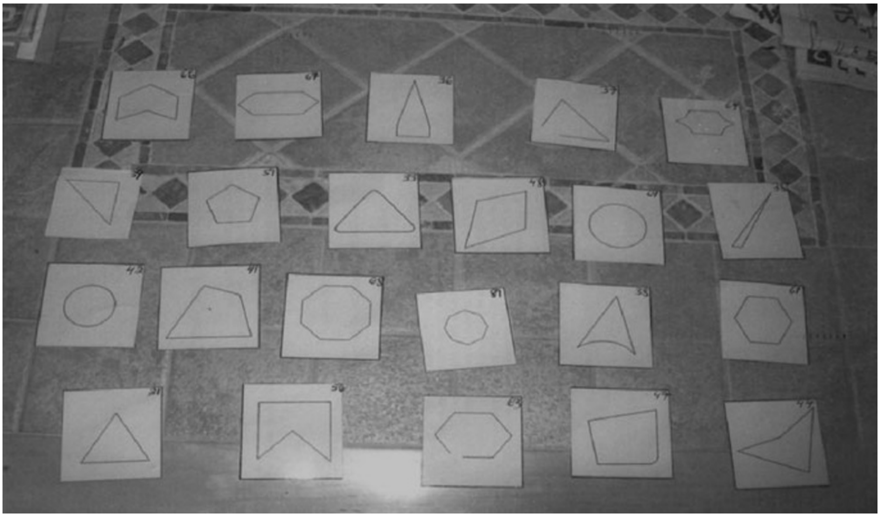


*Figure 12: A mix of 22 shapes.*

Karen (the teacher): Is there a hexagon here?

Tracy: Yes.

Karen: Can you point to it?

Tracy points to a hexagon.

Karen: Is there another hexagon here?

Tracy: Yes. (Tracy points to a different hexagon.)

Karen: Is there another hexagon?

Tracy: Yes. (Tracy points to an octagon.)

Karen: Is there another hexagon?

Tracy: Yes. (Tracy points to the third hexagon.)

Karen: Is there another hexagon?

Tracy: Yes. (Tracy points to the second octagon.)

Karen: Is there another hexagon?

Tracy: No.

What can we learn about Tracy? First, she identified correctly all of the triangles, quadrilaterals, and pentagons. Second, she did not point to any figures which were open, had rounded corners, or curved sides. So, what happened when it came to identifying the hexagons? How come Tracy incorrectly pointed to octagons in addition to the hexagons? If she had pointed to all of the hexagons and only afterwards pointed to the octagons, we may have concluded that Tracy possibly does know that a hexagon must have six sides but felt that she must, at some time, point to all of the polygons. However, this was not the case. She did not "use up" the hexagons and then point to the octagons. It is possible that Tracy did not remember that a hexagon must have six and only six sides and thus included the octagons as well. On the other hand, Tracy did not pause once, after being asked to identify a hexagon, to count vertices. Did she not notice that the octagons have a different number of vertices than the hexagons? Perhaps having only two more vertices is not that big a difference and Tracy just figured that any polygon left must also be a sought after hexagon. Perhaps, since Tracy did not learn to name octagons, it did not occur to her that there were other polygons with more than six sides. Of course, in order to answer these questions, the teacher needs to investigate the matter further. What do you recommend Karen to do? Would teaching Tracy about octagons solve the problem?

## 5.3 SUMMARY

In this chapter we took a critical look at how a geometrical task may be implemented with one child at a time and how a task may be use to both promote knowledge as well as to assess knowledge. Implementing tasks with an individual child has many advantages. Among them, as was illustrated in this chapter, is the chance to provide individual feedback specifically tailored to a child. On the other hand, group activities or tasks provide for interaction, communication, and collective learning. Many tasks implemented with an individual may be adapted for group activity. In doing so, one ought to take into consideration what may be lost, as well as might be gained.

Teachers are ultimately the ones who have the last say regarding which tasks will be implemented in the class as well as how these tasks will be implemented. In the next chapter we present a variety of tasks that were implemented by kindergarten teachers who had participated in our professional development program, *Starting Right: Mathematics in the Preschools.*

CHAPTER 6

# GEOMETRICAL TASKS IN PRESCHOOL: THE VOICE OF THE TEACHER

After having discussed elements of task design, after having reviewed examples of tasks in national guidelines, after having analyzed in great depth two different geometrical tasks, we present in this chapter a variety of tasks which teachers reported implementing in their preschool classes. As you review these tasks, you may ask yourself: What are the affordances and constraints of each task? How may a task be varied to meet the needs of different children? Can the task be used to both promote and evaluate children's knowledge? Would you implement such a task with young children? Why?

## 6.1 EARLY PRESCHOOL

Orna participated in our professional development course for preschool and kindergarten teachers teaching 4-6 year old children. She attended despite the fact that at the time of the course she was teaching 2-3 year olds in a private day care center. Throughout the course, Orna was aware that the children in her care may not be able to reach the same van Hiele levels of geometric reasoning as older children but nevertheless believed that early preparation would benefit even the youngest of her charges. At the end of the course, as did all of the teachers, Orna handed in to the instructors a summary of the mathematical tasks she implemented in her class as well as descriptions of children engaging in the task.

Two of the tasks described by Orna aimed to teach the children about vertices and sides. For the first task, Orna used a ruler to draw a large triangle on a piece of construction paper. Each child received one such drawing. She then provided the children with small round gold stickers. The children's task was to place one gold sticker on each vertex and then count how many stickers they had placed on the paper. For the second part of this task, Orna provided the children with small round orange stickers. The children's task was to then place the orange stickers on the sides of the triangle. The choice of using two different colored stickers was not random. It allowed Orna to emphasize that both attributes of a triangle are important yet distinct. By counting the gold stickers, the attribute of three was emphasized. Orna described the children's engagement as follows:

> The children worked enthusiastically. Each child place a gold sticker exactly on a vertex, except for Miri, who is the youngest child in the class and also the newest. It was a little difficult for them to place the stickers on the sides of the triangle. When I saw that Rachel was getting tired, I drew dots on the sides for her to place one circle on each dot. Shai got tired and walked away.

> When Amy finished, she asked if she could fill in the triangle with extra gold stickers and I allowed her to do so. She was so proud of her creation that she went all around the classroom showing her triangle to everyone.

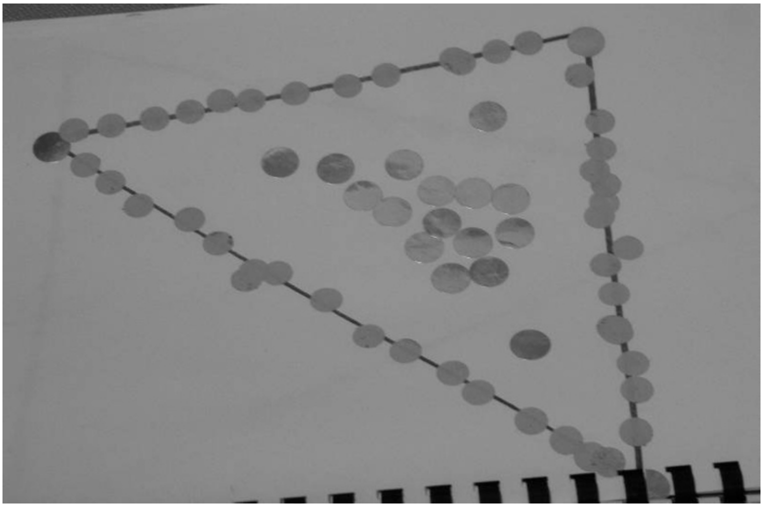


*Figure 1. Gold stickers on the vertices and orange stickers on the sides.*

The second task was designed to be implemented during circle time. In preparation of this task, Orna had prepared various large cut outs of triangles made from corrugated plastic sheets. With all of the children sitting around in a circle, Orna placed these cutouts on the floor in the middle of the circle. The children's task was to walk around the triangles, without stepping on them, to the beat of Orna's tambourine. When the beat stopped, Orna called out either "vertices" or "sides" and the children's task was to stand on either a vertex or side according to what Orna called out. Orna described the children's engagement with this task:

> The children greatly enjoyed this task and knew to stand correctly either on a vertex or a side. Oren tried to put one foot on each vertex and one hand on the third vertex. This task also allowed the children to walk around the triangles, viewing them from different perspectives. Thus the children were able to expand their concept image of a triangle.

Orna also designed a task utilizing the large three-dimensional solids she had in her "climbing corner" of the class. These solids were actually made out of a sponge material and were of the large variety that children could actually climb on. They included tall thin cylinders (where the height was much greater than the diameter), short fat cylinders (where the diameter was much greater than the height), triangular prisms, rectangular prisms, and cubes. The children's task was to build a structure using the solids which Orna named. Orna describes the children as they engage in the task:

*Figure 2: Standing on the vertices*

> I began with cylinders. Amy said that she liked cylinders and went to collect all of the cylinders. She even took the flat pillow-like cylinder. Then I called out prisms and Ron went and got the rectangular prisms. Ortal said that she liked triangles and went and got all the triangular prisms.

As Orna describes the tasks she implemented with young children, she not only details the materials used and the instructions necessary for implementing the task. She is sensitive to and notes the children's difficulties as well as their joy and enthusiasm as they engage in the task. Small children are still developing their fine motor skills. Placing stickers on a straight line is tiring. Children at this age may also have difficulties enunciating complex syllables. Perhaps for this reason, it is Orna who calls out the words vertex and side as well as the names of the three-dimensional figures. Perhaps Orna believes that at this age geometry is something to do and that discussing it might be too difficult.

In the next section we describe additional tasks suggested by preschool teachers teaching 4-6 year old children. As noted above, many of the tasks were tried out during the year, assessed, and went through revisions as the teachers gained experience. It is worthy to note that some of the teachers described implementing identical tasks with their 4-6 year olds as Orna described above. This makes sense. The above tasks are by no means only suitable for very young children. We can imagine that 5 year olds may also learn to differentiate between vertices and sides by placing different color stickers on each attribute. Perhaps the fine motor skills of older children allow them to follow a straight line with more accuracy than a 3-year old can. In the next section we describe additional tasks that teachers suggested implementing with 4-6 year old children.

## 6.2 ADDITIONAL TASKS DESCRIBED BY PRESCHOOL TEACHERS

Many of the tasks implemented by the teachers were first discussed during the course they attended together. For example, in one of the sessions, teachers discussed the difference between creating triangles and identifying triangles. If the task was to create a triangle, then concrete materials needed to be considered.

Ellen: Maybe we can have the children draw triangles or give them a cutout of a triangle and have them trace it.

Instructor: It's probably better to have children trace a cutout. It would be very difficult for them to draw a straight line on their own.

Joy: Or the children can construct triangles from rubber-bands on a geo-board.

Instructor: What would be the advantages and disadvantages of using a material that bends and stretches?

Joy: The line might end up curved.

Instructor: Ok. So, it might be a problem because we want the lines to be straight.

At the end of the professional development course, each teacher summarized the different geometrical tasks she had implemented in her class. In this section, we present a few of the tasks mentioned by different teachers. For each task, the teachers commented on whether it was meant to be implemented in a whole class setting, a small group gathering, or if it was meant for individual engagement. In addition, teachers thought of tasks to be implemented inside the classroom as well as outside in the yard.

*Fishing triangles* – Ellen described this task as one which could be implemented by an individual or by a small group of children. Inside a box there are cutouts of examples and nonexamples of triangles. Each cutout has a magnet stuck on the reverse side. With the help of a magnetized "fishing rod" each child in his turn fishes out a figure, has to tell the group whether or not the figure is a triangle and why, and finally place it in either the pail of triangles or the pail of non-triangles. The children looking on have to approve of the action taken. Of course, this task may be played out with pentagons instead of triangles, and so on.

At the heart of the fishing task is children's ability to sort out examples from nonexamples of whichever figure is the current focus. Many of the teachers described similar tasks implemented with different materials. For example, instead of using cutouts, Hanna drew examples and nonexamples of triangles on cards. This afforded her the opportunity to include the nonexample of an "open triangle" which cannot be demonstrated using cutouts. Instead of magnets, Hanna placed Velcro tape on the backs of the cards. The task was then to sort the figures by placing all of the triangles under the triangle sign on the activity board and all of the non-triangles on the other side of the board under the not-a-triangle sign (see Figure 3). Hanna described this task as one which may be implemented by an individual child or by a pair of children. In the case of the individual, Hanna added that the task would include the child having to explain the sorting to the teacher. When implemented by a pair of children, the explaining would take place between the children.

*Creating a picture out of triangles* – According to Ellen, the aim of this task was to expand children's example space of triangles. The child is given a piece of paper

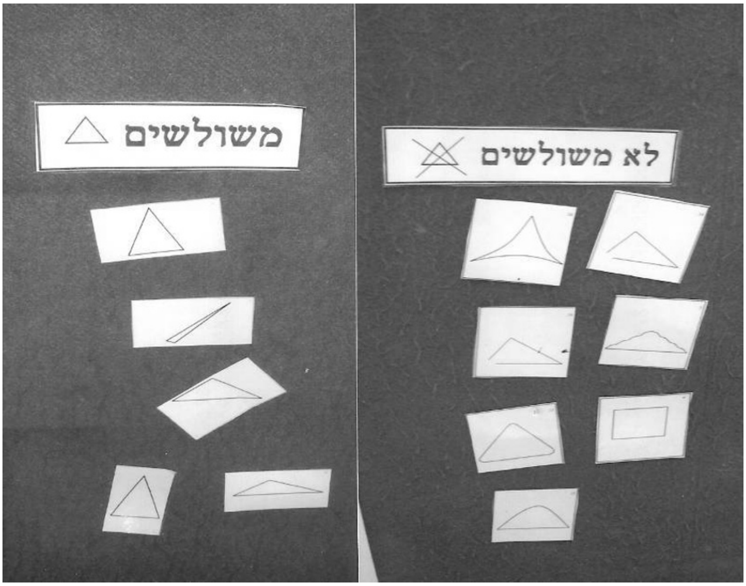



*Figure 3: Triangles and non-triangles.*

with a drawing made up of triangles. In a basket there are many different colored cutouts of triangles which fit the triangles in the picture. The task is to find cutout triangles which match the drawn triangles and construct the same picture from the cut outs.

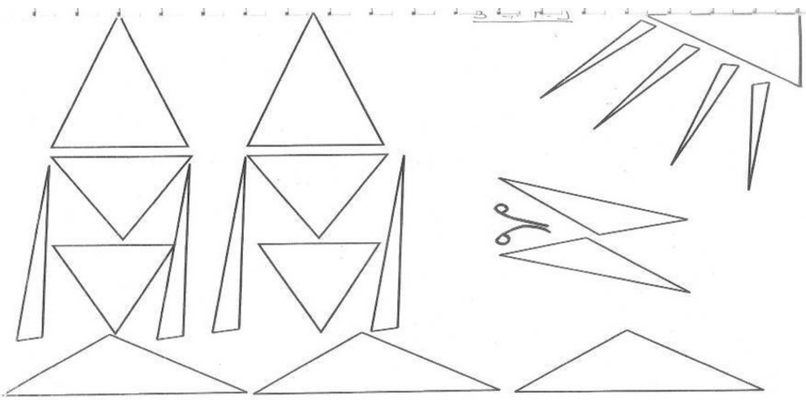


*Figure 4. Ellen's drawing made out of triangles.*

In a variation of this task, Joy produced the picture in Figure 5 and had children color in various shapes according to her instructions. The first time she implemented this task during the year, she instructed children to color in all the triangles using a green marker, the circles with a red marker, and the quadrilaterals with a blue marker. The children, however, refused to comply because, according to their sensibilities, the sun ought to be yellow. They insisted on coloring in all the parts of the sun, the circle and the triangles, in yellow. Thus, at the end of the year, Joy revised her instructions. First, she had the children color in only the triangles in any color which they chose. After she was satisfied that they had indeed done this, she asked them to color in all the quadrilaterals. Finally, the children colored in all the circles. The end result was a multi-colored picture. On the one hand, just from looking at the colors, one would not be able to tell if the child differentiated between the shapes. On the other hand, a triangle or quadrilateral is not determined by its color.

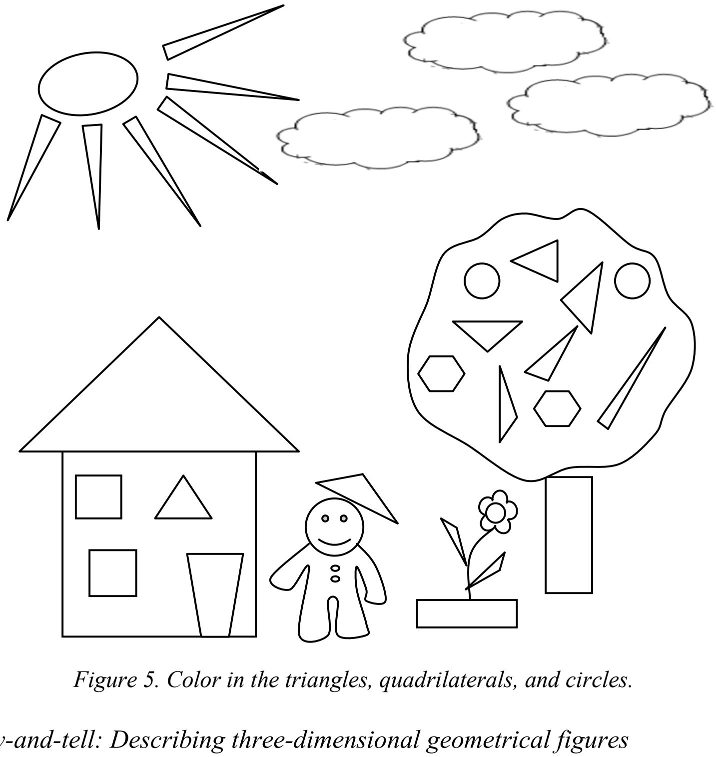


*Figure 5. Color in the triangles, quadrilaterals, and circles.*

*Show-and-tell: Describing three-dimensional geometrical figures*

Lily described this task to be implemented during circle time. The teacher places inside a paper bag an assortment of three-dimensional figures such as balls, cones, cylinders, cubes, and a variety of pyramids. Each child takes a turn pulling out a figure, naming it, showing it to everyone in the class, and describing how it feels and the movement which it can do. For example, a ball feels round all around and can roll in all directions. A cylinder has two “flat” ends and can be rolled forward and backwards. A cone has one “point” and “spins” on its head.

Joy described a similar task also to be implemented during circle time. The teacher spreads out in the middle of the circle many different three-dimensional figures such as those mentioned above. Then she turns to one child and describes the un-named figure which the child must then bring to the teacher. For example, the teacher may request a solid which is made up of squares all around. The child would then bring the teacher a cube. Or the teacher could ask for a solid that has two circles, one on either end and can roll back and forth. The child would then bring the teacher a cylinder.

The above is just a sample of the variety of tasks teachers designed and implemented in their classes during the year. Many teachers described implementing some of the tasks above in the yard. Others described how they implemented some of the tasks with individual children in order to assess a child’s knowledge. In general, some of the task variations stemmed from the teachers’ preferences. Some teachers were more inclined to work with individual children while others leaned towards group activities. Different teachers chose different materials depending on what was available to them as well as what they felt the children would enjoy working with. The plethora of ideas generated by the teachers allowed also for variety within the same kindergarten. Thus, a sorting activity, which aims for children to differentiate between examples and nonexamples of some specific shape, could be implemented once on an activity board, then later on with a fishing game, and again at another time, by coloring in only triangles in a picture made up of triangles and non-triangles. In other words, the variety of tasks implemented, while dependent on the teacher’s choice, still allowed the children to review geometrical concepts in a variety of manners. It is important to note and recall that these teachers had participated in professional development which afforded them to opportunity to enhance not only their geometrical knowledge but their geometrical pedagogical knowledge. We now turn to Part Three, professional development for preschool teachers.

PART 3

# GETTING READY TO TEACH GEOMETRY IN THE PRESCHOOL – PRESCHOOL TEACHER EDUCATION

Would you be surprised to know that many preschool teachers are not required to learn mathematics courses towards their early childhood education degree? Does it seem appropriate? Ginsburg (2008), in his position paper regarding preschool mathematics education reported that typically, early childhood (approximately ages 3-5) educators are "poorly trained to teach the subject, are afraid of it, feel it is not important to teach, and typically teach it badly or not at all" (p. 3). If you are an early childhood teacher educator, have you given thought to what future preschool teachers need to know if they are to introduce geometry and support geometric thinking in their young charges? This part of the book begins by reviewing position and policy papers which outline the roles of the teacher in fostering mathematical and geometrical knowledge in preschool. From these roles, we may derive and theorize about the knowledge that teachers need in order to fulfill these roles. It continues by reviewing past research on preschool teachers' knowledge for teaching mathematics and geometry and then offers a tool for conceptualizing preschool teachers' knowledge for teaching geometry. The second chapter of this part illustrates how preschool teachers' knowledge for teacher geometry may be enhanced with professional development. Finally, we offer some tasks that may be implemented in the professional development of preschool teachers.

CHAPTER 7

# CONCEPTUALIZING PRESCHOOL TEACHERS' KNOWLEDGE FOR TEACHING GEOMETRY

## 7.1 POSITION PAPERS, POLICY REPORTS, AND NATIONAL GUIDELINES: WHAT DO THEY RECOMMEND?

Who says that preschool teachers need to teach geometry (or for that matter any mathematics)?!? In the past, researchers held the opinion that young children have little knowledge of mathematics and should not begin learning mathematics before beginning formal schooling in elementary school (Bereiter & Engelmann, 1966; Thorndike, 1922). Recently, a joint position paper published in the United States by the National Association for the Education of Young Children (NAEYC) and the National Council for Teachers of Mathematics (NCTM) stated that "high quality, challenging, and accessible mathematics education for 3- to 6-year-old children is a vital foundation for future mathematics learning" (NAEYC & NCTM, 2002, p.1). As such, they put forth 10 research-based recommendations to guide classroom practice as well as four recommendations for policies, system changes, and other actions essential for the support of these practices (see Figure 1). While reading the guidelines for classroom practice, ask yourself: What knowledge must a preschool teacher have in order to fulfill the many roles outlined in the position paper?

**In high-quality mathematics education for 3- to 6-year-old children, teachers and other key professionals should**

1. enhance children's natural interest in mathematics and their disposition to use it to make sense of their physical and social worlds

2. build on children's experience and knowledge, including their family, linguistic, cultural, and community backgrounds; their individual approaches to learning; and their informal knowledge

3. base mathematics curriculum and teaching practices on knowledge of young children's cognitive, linguistic, physical, and social-emotional development

4. use curriculum and teaching practices that strengthen children's problem-solving and reasoning processes as well as representing, communicating, and connecting mathematical ideas

5. ensure that the curriculum is coherent and compatible with known relationships and sequences of important mathematical ideas

6. provide for children's deep and sustained interaction with key mathematical ideas

7. integrate mathematics with other activities and other activities with mathematics

8. provide ample time, materials, and teacher support for children to engage in play, a context in which they explore and manipulate mathematical ideas with keen interest

9. actively introduce mathematical concepts, methods, and language through a range of appropriate experiences and teaching strategies

10. support children's learning by thoughtfully and continually assessing all children's mathematical knowledge, skills, and strategies.

**To support high-quality mathematics education, institutions, program developers, and policymakers should**

1. create more effective early childhood teacher preparation and continuing professional development

2. use collaborative processes to develop well-aligned systems of appropriate high-quality standards, curriculum, and assessment

3. design institutional structures and policies that support teachers' ongoing learning, teamwork, and planning

4. provide resources necessary to overcome the barriers to young children's mathematical proficiency at the classroom, community, institutional, and system-wide levels.

*Figure 1. NAEYC and NCTM (2002) guidelines for preschool mathematics.*

As can be seen from the 10 recommendations which guide classroom practice, the teacher has a vital, active, and multi-faceted role in promoting children's mathematical knowledge. She must take into consideration children's past experiences and knowledge, be able to teach mathematical concepts in a variety of ways, and encourage children's mathematical reasoning, language, and communication skills. Did you think about the knowledge a preschool teacher would need in order to carry out the 10 recommendations listed above? It seems obvious that if the teacher is to promote mathematical learning, she must know mathematics. Yet, only a few of the recommendations seem specifically related to a teacher's mathematical knowledge. For example, Point 9 specifically mentions that the teacher should introduce mathematical concepts, methods, and language. Therefore, we may infer that the teacher must know the mathematical concepts, methods, and language which are to be taught at this age. However, Point 2 encourages the teacher to consider the student's experiences, cultural and familiar backgrounds, and informal knowledge. How may Point 2 be related to teachers' knowledge for teaching mathematics? Point 8 advises teachers to provide materials and a context in which children may explore mathematical ideas. What knowledge would support teachers in their endeavor to provide appropriate materials and create a supportive environment for learning mathematics? Is knowing about number concepts and shapes sufficient for being able to teach geometry in preschool?

Although it was not the intention of the position statement to specifically outline content, geometry is mentioned in several places. At times, learning geometrical concepts is the specific goal. At other times, knowing geometrical concepts is mentioned as a means to achieving other goals. For example, when elaborating on how teachers should take into consideration individual approaches to learning, the paper states that some children "learn especially well when instructional materials and strategies use geometry to convey number concepts" (p. 4). How is this related to teachers' knowledge? Basically, the preschool teacher must not only know geometrical concepts in order to teach geometry but must also know how to use geometrical concepts to teach number conceptions. Another role of the preschool teacher is to make connections between mathematical ideas. Here too, geometry is mentioned specifically. "When children connect number and geometry (for example, by counting the sides of shapes, using arrays to understand number combinations, or measuring the length of their classroom), they strengthen concepts from both areas and build knowledge and beliefs about mathematics as a coherent system" (p. 6). Again, it is not enough for the teacher to know how to teach number concepts and geometry concepts. The teacher must also know how to connect concepts from both mathematical domains. More recently, the Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics (NCTM, 2006) identified curriculum focal points for prekindergarten and kindergarten children, specifically mentioning geometry as an emphasized topic for children of this age. Children should be able to identify and describe a variety of two- and three-dimensional shapes presented in a variety of ways and use geometrical concepts when recognizing working on simple sequential patterns or when analyzing a data set. For example, the attributes children identified in relation to geometry may be used to analyze a set of data objects.

The Australian Association of Mathematics Teachers (AAMT) and Early Childhood Australia (ECA) also published a joint position paper addressing the importance of early childhood mathematics. This paper listed no less that 16 recommendations of pedagogical practices for early childhood educators to adopt. These included encouraging young children to see themselves as mathematicians, focusing on the use of mathematical language to describe and explain mathematical ideas, and assessing young children's mathematical development (AAMT / ECA, 2006). Unlike the position paper published in the US, the Australian paper makes no mention of specific mathematical content. And yet, we can derive from the few recommendations mentioned above that if a preschool teacher is going to promote geometrical knowledge, she must herself know appropriate geometric language, be able to explain geometrical concepts, and assess her young students' geometrical knowledge. In England, the Statutory Framework for the Early Years Foundation Stage (2008) states that "children must be supported in developing their understanding of Problems Solving, Reasoning and Numeracy in a broad range of contexts…They must be provided with opportunities to practice and extend their skill… to gain confidence and competence in their use" (p. 14). The framework goes on to list 12 early learning goals that children should reach by the age of five. When assessing the achievement of these goals, geometry is related to in that practitioners should check if the child "uses mathematical language to describe solid (3D) objects and flat (2D) shapes" (p. 47). Reference to the preschool teacher's roles in achieving these goals is made in the accompanying non-statutory Practice Guidance for the Early Years Foundation Stage (2008). There, it is suggested that practitioners attend to three major areas: positive relationships, enabling environments, and learning and development. The first area includes giving children sufficient time and space to use new mathematical ideas during child-initiated activities, encouraging children to explore real-life problems, supporting children in their communication of ideas other than spoken English, and valuing children's own explorations. The second area includes recognizing the potential of both outdoor and indoor environments to explore mathematical concepts. The third area includes supporting the learning and development of mathematical ideas through a variety of activities such as stories, songs, and games.

The mandatory Israel National Mathematics Preschool Curriculum (INMPC, 2008), also recognizes the many roles of the preschool teacher in the mathematical development of the children.

> The mathematical development of the child is related to the opportunities the child has to engage in mathematics in preschool, to the way in which the preschool teacher exposes the child to mathematics, to the type of activities and tasks which the teacher presents to the child, to the ability of the teacher to follow the child's development and advance this development … Young children also learn by mimicking the teacher. Therefore, it is important for the preschool teacher to use correct mathematical language when speaking to the children so that they may become used to mathematical language and repeat it. The use of correct mathematical language may prevent or minimize misconceptions later on. (INMPC, p. 15)

Throughout the curriculum guidelines, there are suggestions for the teacher on how to plan explicit mathematical activities as well as how to take advantage of mathematical activities which may spontaneously arise in the class.

To summarize, teachers' knowledge must be sufficient in order to specifically teach geometrical concepts as well as to use geometrical concepts in order to reach more global mathematical aims. Taking into consideration the many position papers as well as curriculum guidelines mentioned above, Table 1 summarizes some of the direct aims of learning geometry as well as some of the indirect ways geometry is used as a tool in the achievement of more global aims.

Take a minute and review once more the recommendations listed in the various position papers and curriculum guidelines. Now attempt to adapt these general recommendations to the teaching of geometrical concepts in preschool. Can you put them all together and conceptualize what the preschool teacher would need to know if she were to attempt to carry out even some of the guidelines?

*Table 1. Some aims of learning geometry.*

| | |
|---|---|
| Geometry as the specific aim | By the end of kindergarten children should be able to: Identify, name, describe, and sort two- and three-dimensional figures according to the attributes of these figures. |
| Geometry as a tool in achieving global aims | Through geometry children learn to develop spatial reasoning skills, discriminate between critical and non-critical attributes of a concept, and learn to connect concepts to their daily lives becoming aware of the difference between every-day language and mathematical language. |

The National Association for the Education of Young Children (NAEYC) and the National Council for Teachers of Mathematics (NCTM) recommend that "teachers of young children should learn the mathematics content that is directly relevant to their professional role" (p. 14). Similarly, the Australian Association of Mathematics Teachers (AAMT) and Early Childhood Australia (ECA) recommended that early childhood staff be provided with "ongoing professional learning that develops their knowledge, skills and confidence in early childhood mathematics" (p. 4). National guidelines recognize the importance of increasing early childhood educators' knowledge for teaching mathematics and geometry but fall short of stating what this knowledge is.

We have begun to formulate some ideas regarding the knowledge a preschool teacher would need. Conceptualizing teacher's knowledge is not a simple or trivial task. In the following section we discuss different theories regarding teachers' knowledge for teaching mathematics and how they relate to knowledge necessary for teaching geometry in preschool.

## 7.2 THEORIES OF TEACHERS' KNOWLEDGE

Suppose you are a kindergarten teacher who would like to introduce the concept of a square in class. What do you need to know in order to teach five-year olds about squares? Obviously, you need to know what a square is. Consider the following figures (See Figures 2a and 2b):

*Figures 2a and 2b. Two squares or a square and a diamond?*

Are they both squares? Isn't the second figure a diamond? Can a shape be both a diamond and a square? As a preschool teacher, you might be called upon to answer these questions. A square may be defined as a quadrilateral with four equal sides and four right angles. Is this how you will define it to your students? Is it the only way to define a square?

Breaking down teachers' knowledge into different components allows us, teachers, teacher educators, and researchers, to examine what knowledge is necessary for teaching a subject or a specific concept. In his seminal work, Shulman (1986) described and analyzed components of teachers' knowledge necessary for teaching. Two of the major components identified were subject-matter knowledge (SMK) and pedagogical content knowledge (PCK). SMK refers to the knowledge of facts, rules, procedures, and concepts required of a specific domain. SMK may be further divided into two levels (Even & Tirosh, 1995): "knowing that" and "knowing why". Thus, regarding the square, it is important to know that some figure is a square as well as be able to explain why that specific figure is a square. SMK also includes understanding the structure of the subject matter. In mathematics, this includes knowing how definitions for the same concept may be equivalent and how definitions may be used to discern between examples and nonexamples of a concept. Thus, while a square may defined as a quadrilateral with four equal sides and four right angles, it may also be defined as a rectangle with two equal adjacent sides.

Being able to draw or describe squares is important but not enough. Teachers also need to be able to explain to their students why some figure is or is not a square. PCK refers to "the ways of representing and formulating the subject that make it comprehensible to others … what makes the learning of specific topics easy or difficult" (Shulman, 1986, p. 9). It includes knowledge of appropriate analogies, examples, illustrations, and explanations. Regarding the square, it includes knowing that children often mistakenly see orientation as a critical attribute (Hannibal, 1999) and thus it is important to present a wide variety of examples of squares to children, including the diamond-like square in Figure 2b.

Shulman's theory of knowledge was further developed by Ball and her colleagues (e.g. Ball, Bass, & Hill, 2004; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008). Basing their theory on the work that mathematics teachers do in the classroom, they divided SMK further into common content knowledge (CCK) and specialized content knowledge (SCK). CCK may be defined as "the mathematical knowledge and skill used in settings other than teaching" (Ball, Thames, & Phelps, 2008, p. 399) whereas SCK is "mathematical knowledge not typically needed for purposes other than teaching" (Ball, Thames, & Phelps, 2008, p. 400).

Within the context of geometry, Ball et al. (2008) gave a few examples illustrating the dimensions of knowledge for teaching in elementary school. Knowing that the diagonals of a parallelogram are not necessarily perpendicular may be considered knowledge typical of anyone who knows mathematics (CCK). Knowing "how the mathematical meaning of edge is different from the everyday reference to the edge of a table" (p. 400) is an example of SCK. You may be tempted to think that any adult knows enough mathematics to teach geometry in preschool. After all, we are not talking about calculus or college mathematics. We are talking about nursery and kindergarten! Let us consider again the square. You know how to draw a square, identify a square, and describe a square. This type of knowledge may be considered CCK. Do you know how squares are related to triangles? Have you thought about how squares are related to rectangles? Have you thought about how squares are related to cubes? This is knowledge more likely to be held by teachers and thus may be considered SCK. Now consider circles. You know how to draw circles (maybe with the help of a compass) and you can identify a circle. But can you explain the difference between a circle and an ellipse? These examples of SCK are in line with guidelines suggested by the various position papers mentioned previously in that they demonstrate the kind of mathematics knowledge preschool teachers must know in order to ensure a cohesive mathematics curriculum where children are able to make connections between mathematical ideas.

Shulman's (1986) theory of pedagogical content knowledge was also further differentiated by Ball et al (2008) into knowledge of content and students (KCS) and knowledge of content and teaching (KCT). KCS is "knowledge that combines knowing about students and knowing about mathematics" whereas KCT "combines knowing about teaching and knowing about mathematics" (Ball, Thames, & Phelps, 2008, p. 401). Consider the triangle and take a look at the shapes in Figure 3.

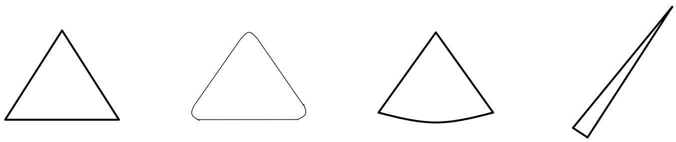


*Figure 3. Which of the figures are triangles?*

Which will children identify as a triangle? How will children explain their reasoning regarding why some figure is a triangle and another is not? Knowing the answers to these questions is the type of knowledge exemplified by KCS. Taking into consideration children's reasoning, the teacher must also know how to present to preschoolers the concept of a triangle. This includes knowing how to sequence the presentation of examples and which examples may deepen students' conceptual knowledge. This is typical of KCT. These aspects of teachers' knowledge,

knowledge of students and knowledge of teaching, were also implied by the position papers mentioned in the previous section. Recall that the various guidelines suggested that preschool teachers take into consideration children's experiences and informal ways of thinking when planning mathematical activities. The guidelines also suggested that teachers engage in teaching practices that encourage mathematical reasoning and problem solving processes. In other words, Ball et al's four dimensions of mathematical knowledge may also be used to describe preschool teachers' knowledge for teaching mathematics.

Alongside theories of teachers' knowledge for teaching mathematics there exist theories of mathematics knowledge. That is, as we consider the knowledge needed for teaching mathematics, we also consider what it means to know mathematics. This is discussed in the next section.

## 7.3 CONCEPT IMAGE/CONCEPT DEFINITION – A THEORY OF MATHEMATICS KNOWLEDGE

There are several theories related to the components of mathematics knowledge. In Part One of this book we mentioned Tall and Vinner's (1981) concept image/concept definition (CICD) theory. We review this theory again in this section and show how it may be used in conjunction with theories of teachers' knowledge to provide a comprehensive theory of preschool teachers' knowledge for teaching geometry.

In general, the CICD theory focuses on the image of a concept as well as and opposed to the definition of a concept. We consider this theory especially appropriate when considering young children's mathematical knowledge as they develop knowledge of mathematical concepts well before they learn formal definitions.

Having precise definitions for mathematical concepts allows for mathematical coherence and provides the foundation for building mathematical theories. In geometry, for example, the definition of a square may be based on the definition of a rectangle, which in turn may be based on the definition of a parallelogram. However, these same mathematical concepts may have been encountered by the individual in other forms prior to being formally defined. The 3-year old may recognize a square but almost certainly has not encountered a precise definition for a square. Even after they are defined, mathematical concepts often invoke images both at the personal as well as the collective level. The term concept image is used to describe "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). The concept definition refers to "a form of words used to specify that concept" (p. 152). A formal concept definition is a definition accepted by the mathematical community whereas a personal concept definition may be formed by the individual and change with time and circumstance. Because the concept image actually contains a conglomerate of ideas, some of these ideas may coincide with the definition while others may not. For example, what image comes to the mind of children when they think of a rectangle? In one study (Clements, Swaminathan, Hannibal, & Sarama, 1999) it was suggested that children have a prototype of a rectangle which is long, for the most part disregarding orientation. Thus, many young children may incorrectly identify a long parallelogram as a rectangle. It was also found that rectangles that were too narrow or not narrow enough were not accepted. These images may clash with the definition of the rectangle.

When a problem is posed to an individual, there are several cognitive paths that may be taken which take into account both the concept image and concept definition. At times, although the individual may have been presented with the definition, this particular path may be bypassed. According to Vinner (1991), an intuitive response is one where "everyday life thought habits take over and the respondent is unaware of the need to consult the formal definition" (p. 73). Intuitive knowledge is both self-evident and immediate and is often derived from experience (Fischbein, 1987). As such it does not always promote the logical and deductive reasoning necessary for developing formal mathematical concepts. "Sometimes, the intuitive background manipulates and hinders the formal interpretation" (Fischbein, 1993, p. 14). As discussed in Part One of this book, Fischbein (1993) considered the figural concepts an especially interesting situation where intuitive and formal aspects interact. The image of the figure promotes an immediate intuitive response. Yet, geometrical concepts are abstract ideas derived from formal definitions. Thus, as we consider the notions of concept image and concept definition, we take into account aspects of Fischbein's theory related to intuitive and formal knowledge.

Although Tall and Vinner (1981) introduced their theory within the context of advanced mathematical thinking, the interplay between concept definition and concept image is part of the process of concept formation at any age. Young children learn about and develop concepts, including geometrical concepts, before they begin kindergarten. As such, their concept image is often limited to their immediate surroundings and experiences and is based on perceptual similarities of examples, also known as characteristic features (in line with Smith, Shoben, & Rips, 1974). When viewing the equilateral triangle in Figure 3, one five-year old commented, "It's a triangle because it looks like the roof a house". Indeed, a non-academic study of children's books on shapes (Haven't we all read such books to our children or grandchildren?) revealed that the triangle was most often associated with the roof a house. These books are meant to be read to young children well before they reach the age of five. This initial discrimination may lead to only partial concept acquisition in that children may consider some nonexamples to be examples and yet may consider some examples to be nonexamples of the concept. Regarding geometrical concept formation, van Hiele (1958) theorized that students' geometrical thinking progresses through a hierarchy of five levels, eventually leading up to formal deductive reasoning. At the most basic level, students' use visual reasoning, taking in the whole shape without considering that the shape is made up of separate components. At the second level students begin to notice the different attributes of different shapes but the attributes are not perceived as being related. At the third level, relationships between attributes are perceived and definitions are meaningful. Kindergarten children begin to perceive attributes but need guidance in order to assess which attributes are critical for identifying a figure and which are not.

After having briefly discussed components of geometrical knowledge, and having previously discussed components of teachers' knowledge for teaching, we now combine these theories into a more coherent framework which can be used to discuss preschool teachers' knowledge for teaching geometry.

## 7.4 COMBINING A THEORY OF TEACHERS' KNOWLEDGE WITH A THEORY OF MATHEMATICS KNOWLEDGE

In framing the knowledge necessary for teaching geometry in preschool we rely on Shulman's (1986) notions of SMK and PCK. We suggest that SMK may be further divided into topic-specific mathematical knowledge (TMK) and general mathematical knowledge (GMK). TMK is knowledge related to a specific topic. For example, knowing the definition of a square is knowledge specifically related to teaching about squares. GMK refers to knowledge that applies to many different mathematical domains and is not specific to only one topic. For example, knowledge of mathematical reasoning, knowing the difference between, for example, deductive and inductive reasoning, is knowledge necessary for teaching a wide range of mathematical topics. We apply Ball's notions of KCS and KCT, as discussed above, in order to differentiate between types of PCK. We suggest that these four dimensions of teachers' knowledge be combined with Tall and Vinner's concept image/concept definition theory in order to provide a finer grain and more focused lens with which to study preschool teachers' knowledge for teaching. Such a framework would allow us to investigate, for example, teachers' knowledge of the psychological aspects of student's mathematical errors. We illustrate this framework by focusing on teachers' knowledge for teaching triangles.

| | Domains of teachers' knowledge | | | |
|---|---|---|---|---|
| Domains of mathematical thinking | TMK | GMK | KCS | KCT |
| Concept image | Cell 1 | Cell 2 | Cell 3 | Cell 4 |
| Concept definition | Cell 5 | Cell 6 | Cell 7 | Cell 8 |

*Figure 4. A combined-theories framework for teachers' knowledge for teaching geometry.*

Cell 1: TMK-Image. Here we address the topic-specific knowledge of a concept's image. Regarding triangles, this includes a rich concept image of triangles which incorporates scalene and obtuse triangles and not just equilateral and isosceles triangles. It may also include a broad image of nonexamples for triangles. For example, what image comes to mind when you think of something that is not a triangle? Tsamir, Tirosh, and Levenson (2008) asked 28 adults with at least a first degree in science, mathematics, or engineering to give an example of "something that is not a triangle". They were then each asked to give another example of "something that is not a triangle". The immediate first response of all the adults was a circle. Their second example was a square (24 adults) or a rectangle (4 adults). Yet teachers' concept image of nonexamples of triangles should also include other nonexamples such as those featured in Figure 5.

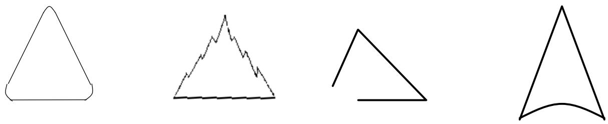


*Figure 5. Some nonexamples of triangles.*

Cell 2: GMK-Image. Here we address the general knowledge of a concept's image necessary for teaching. This includes knowing that, in general, orientation of a figure or the thickness of a line should not play a factor when determining two-dimensional geometrical figures.

Cell 3: KCS-Image. Here we address knowledge related to students and concept images. This includes knowing that the equilateral triangle is a prototypical triangle (Hershkowitz, 1990) and that young children may not identify as a triangle a long and narrow triangle such as the scalene triangle, even when admitting that it has three points and lines (Shaughnessy & Burger, 1985). It also includes knowing children's concept image of nonexamples. For example, Tsamir, Tirosh, and Levenson (2008) asked 22 kindergarten children to give an example of something that is not a triangle. The immediate first response of 18 children was a circle. The four others said a square.

We also include in this cell knowledge of the van Hiele model (e.g., van Hiele & van Hiele, 1958) for students' geometrical thinking and being able to recognize, for example, that a student's concept image at the most basic level takes in the whole shape without considering its components. As such, this cell includes knowing that a rounded "triangle" such as the first shape in Figure 5, is often identified as a triangle (Hasegawa, 1997) because children take in the likeness of the whole shape, ignoring that the shape is missing vertices.

Cell 4: KCT-Image. Here we address knowledge related to teaching and concept images. This includes knowing which examples and nonexamples to present to a student which will broaden his concept image of a triangle to include, for example, triangles with different orientations (see Figure 6).

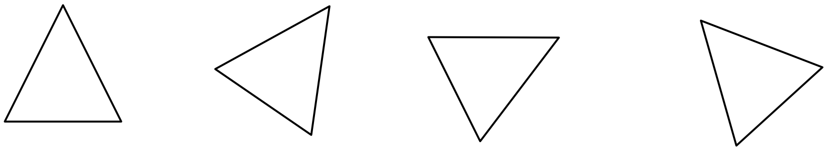


*Figure 6. The same triangle illustrated with different orientations.*

Cell 5: TMK-Definition. Here we address topic-specific knowledge related to a concept's definition. It includes knowing one or more definitions for a triangle. It also includes knowing that defining the triangle as a three-sided polygon implies that it must be a closed figure with three vertices. It includes knowing that the triangle may be defined as a three-sided polygon, or a polygon with three angles, or a polygon with three vertices and that all three definitions are equivalent.

Cell 6: GMK-Definition. Here we address general knowledge of definitions. In mathematics, definitions are apt to contain only necessary and sufficient conditions required to identify an example of the concept. Other critical attributes may be reasoned out from the definition. This applies to definitions of triangles as well as to definitions of other two-dimensional shapes, as well as to definitions of non-geometric mathematical concepts.

Cell 7: KCS-Definition. Here we address knowledge related to students and concept definitions. It includes knowing that a minimalist definition may not be appropriate for young students at the first or second van Hiele level because they do not necessary perceive that a polygon with three sides must necessarily have three vertices. For example, research has suggested that for young children, the association between a triangle and the attribute of threeness may be stronger than the necessity for it to be closed or for its vertices to be pointed (Tsamir, Tirosh, & Levenson, 2008). Some children who identified the rounded "triangle" (the first shape in Figure 5) as a triangle claimed, "it has three corners even though it's rounded." Referring to the zig-zag "triangle" (the second shape in Figure 5) one child claimed it was a triangle "even though it has a lot of points."

Cell 8: KCT-Definition. Here we address knowledge related to teaching and concept definitions. It includes speaking to children with precise language, calling the vertices of a triangle by their proper name as opposed to referring to them as corners. In fact, many of the position papers mentioned in the first section of this chapter pointed to the importance of using mathematical language and encouraging children to use mathematical terms during play and daily routines. Knowledge in this cell also includes knowing which examples and nonexamples of a triangle to present to children which may encourage children's use of concept definitions and promote their advancement along the van Hiele levels of geometrical thinking. For example, the following nonexamaples (see Figure 7) do not necessarily encourage children to refer to critical attributes when reasoning about triangles (Tsamir, Tirosh, & Levenson, 2008).

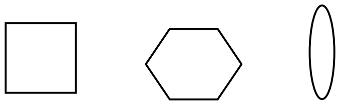


*Figure 7. Intuitively recognized non-triangles.*

Instead, these figures were found to be intuitively recognized as non-triangles in that children recognized them immediately without feeling the need to justify their identification. On the other hand, the presentation of nonexamples of a triangle which are not intuitively recognized as such, such as those shown in Figure 8, may encourage children to refer back to the concept definition when identifying the figure as a nonexample of a triangle (Tsamir, Tirosh, & Levenson, 2008).

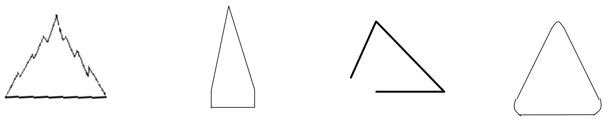


*Figure 8. Non-intuitive non-triangles.*

## 7.5 SUMMING UP

There are many mathematical knowledge theories which could be presented to teachers. Choosing which ones are particularly relevant to teaching and which ones to present to teachers can be complex (Tsamir, 2008). Regarding mathematics teaching, the concept image concept definition theory is a widely recognized mathematics education theory which spans students of all ages and is relevant to many different mathematical contexts (Hershkowitz, 1989; Schwarz & Hershkowitz, 1999; Tall & Vinner, 1981; Vinner & Dreyfus, 1989). It informs our understanding of mathematical concept formation. It allows us to predict and analyze students' errors. Another direction for widening the use of the tool would be to consider combining other theories of mathematical knowledge with theories of teachers' knowledge. Tsamir (2008) described how familiarizing secondary school teachers' with Fischbein's (1993) theory of the three components of knowledge and Stavy and Tirosh's (1996) theory of the intuitive rules may promote secondary teachers' mathematical and pedagogical knowledge. The choice of theories may depend on the mathematical context as well as the activities or tasks which take place in the classroom. For example, parts of the intuitive rules theory are especially appropriate when engaging in comparison tasks. Combining the intuitive rules theory with the four dimensions of teachers' knowledge may then focus us, for example, on developing teachers' knowledge of how and when students' use these rules (KCS). Another direction for addressing this issue might be to pool mathematical education theories that investigate students' mathematical learning and possible sources of errors. For example, Fischbein's (1993) theory mentioned above, the intuitive rules theory (Stavy & Tirosh, 1996), and Tall and Vinner's (1981) concept image/concept definition theory all have elements of intuitive thinking. The mathematics education research community could consider how to combine these theories in order to provide a more comprehensive theory for investigating students' mathematical thinking as well as teachers' mathematics knowledge for teaching.

The combined theory we have suggested in this chapter addresses many of the issues stipulated in the various position papers and guidelines mentioned in the first section. The teacher has many roles. She must take into consideration children's prior knowledge, intuitions, development, and cultural background as she plans appropriate activities. For this it is important to be knowledgeable of content and students. She is advised to plan activities which focus on mathematics as well as know how to take advantage of every day occurrences which can be used as impetus to teach mathematics. For this she needs knowledge of content and teaching. She is encouraged to use mathematical language and encourage her children to use mathematical language. This is subject matter knowledge for teachers. And of course, she must know how to break down the geometry and how geometrical knowledge is developed. For this, the concept image concept definition theory is particularly appropriate.

The theoretical framework described here may be used to build teachers' knowledge in at least two ways. First, it serves as a tool for teacher educators by allowing the teacher educator to focus on the specific knowledge being promoted. In much the same way, when explicitly presented to teachers it may also serve to focus the teachers on the knowledge they are building and its use in teaching. In the next chapter we discuss professional development for preschool teachers which aims to enhance teachers' knowledge for teaching geometry using the theoretical framework described here.

CHAPTER 8

# ENHANCING PRESCHOOL TEACHERS' KNOWLEDGE FOR TEACHING MATHEMATICS

Professional development for practicing teachers may vary in duration, form, and content. A program attended by teachers may range from a one-day summer meeting followed by eight workshops during the year, to a semester course given on a weekly basis (Tsamir, 2008), to an intensive two-year program (Graven, 2004). The program may take the form of university courses immersed in theory or workshops immersed in practice. Relating to early childhood teachers, the NAEYC and NCYM (2002) offered some guidelines: "Inservice professional development needs to move beyond the one-time workshop to deeper exploration of key mathematical topics as they connect with young children's thinking and with classroom practices" (NAEYC & NCTM, 2002, p. 6). In the previous chapter we discussed a framework for preschool teachers' knowledge which takes into consideration both theories of teachers' knowledge as well as theories of mathematical knowledge. In this chapter we illustrate how preschool teachers' knowledge for teaching geometry may be enhanced with professional development.

## 8.1 BACKGROUND

From the year 2006 we have continuously provided professional development for groups of preschool teachers with various educational backgrounds. Our program, Starting Right: Mathematics in Preschools, carried out in collaboration with the Rashi Foundation and the Israel Ministry of Education, was aimed at teachers who had a college degree in early childhood education and a state certified license to teach 4-6 year old children in state-run nursery and kindergarten classes. Over a two-year period, teachers met with the instructors on a weekly basis (4 hours per week). During the first year of the program, on a rotational basis, sessions took place at one of the preschool classrooms. During the second year of the program, sessions took place either at a local educational center or in one of the preschools. Each session was video-recorded and transcribed.

Throughout this program, each teacher was visited by a member of the professional development staff who came to the preschool on a weekly basis, sitting with children and guiding the teacher in her endeavor to create a mathematically enriched environment for her children. Attention was given to both teachers' and children's affect. The aim of the program was for all children to exhibit competency in all mathematical areas taught according to the mandatory national curriculum for these ages (such as geometry, numbers, and operations).

Our program, First Steps in Mathematics, carried out in collaboration with the WIZO foundation day care centers, was aimed at practitioners and caregivers who did not necessarily have a college degree in early childhood education.[1] These practitioners worked in daycare centers caring for children up until and including the age of three. Over a two year period, ten workshops took place aimed at increasing practitioners' mathematical and pedagogical knowledge. In addition, practitioners observed program staff members as they engaged young children in mathematical activities.

In the previous chapter we laid out a framework for describing preschool teacher's knowledge for teaching geometry. The framework combined elements of Shulman's (1986) and Ball et al's (2008) components of knowledge for teaching with Tall and Vinner's (1981) concept image concept definition theory of mathematics knowledge. Shulman's notion of SMK was differentiated into topic-specific mathematical knowledge (TMK) and general mathematics knowledge (GMK). We also incorporated Ball's notions of knowledge of content and students (KCS) and knowledge of content and teaching (KCT), refinements of Shulman's (1986) PCK. In the following sections we describe different segments of our professional development programs, which illustrate how the combined theory framework suggested in the previous chapter may be used to build and assess preschool teachers' knowledge for teaching geometry.

## 8.2 PROGRAM SEGMENTS

In order to differentiate between the participants of the different programs, we note that preschool teachers refer to the group of nursery and kindergarten teachers who participated in the program Starting Right: Mathematics in Preschools whereas practitioners refer to the group of caregivers who participated in the program First Steps in Mathematics.

### *Assessing preschool teachers' TMK and GMK regarding concept images*

All of our programs for preschool teachers began with the topic of triangles. We began with triangles for several reasons. First, the preschool curriculums in many countries, including England, the U.S., and Israel specify that preschool children should be able to recognize many different examples of triangles. Second, we hypothesized that preschool teachers would be somewhat familiar with the definition of a triangle and perhaps less familiar with definitions for other geometrical figures. For example, although squares and rectangles may be familiar to many, the hierarchical nature of quadrilaterals, make the square a complex figure to define (De Villiers, 1994).

What image comes to mind when one thinks about a triangle and what TMK and GMK must teachers know regarding this concept image? Recall that according to Tall and Vinner (1981) the concept image consists of mental images, properties,

[1] This program was supported by the WIZO foundation in Frankfurt, Germany.

and processes associated with the concept. A concept image may also change with time and experience. Studies have shown that when asked to draw a triangle, most people will draw either an equilateral triangle or isosceles triangle with a horizontal base (Hershkowitz, 1990). Thus, our first task was to assess teachers' concept image of triangles.

During our first meeting with the preschool teachers, we asked the teachers to draw three examples of triangles and three nonexamples of triangles. First, we note that the examples teachers drew were indeed triangles and all of the nonexamples teachers drew were indeed not triangles. In other words, the teachers demonstrated TMK of the concept image for a triangle. Eight of the nine teachers present during this session drew at least one example of an equilateral triangle with a horizontal base. The ninth teacher drew a triangle with unequal sides but with a horizontal base. Five teachers drew only equilateral triangles with horizontal bases for all three examples of triangles (see Figure 1). At this point we were unsure if the teachers' general knowledge of a triangle's concept image included knowing that orientation may be varied.

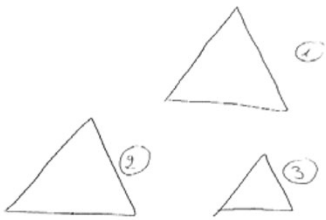


*Figure 1. One teacher's response to the task of drawing three examples of triangles.*

Two teachers drew one right triangle each, with horizontal bases. Only three teachers drew examples of triangles with a different orientation (see Figure 2).

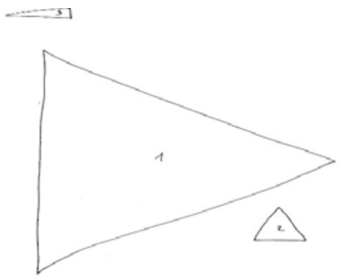


*Figure 2. A second teacher's response to the task of drawing three examples of triangles.*

Regarding the nonexamples, all of the teachers drew geometrical figures such as circles, squares, and trapezoids (see Figure 3). These results indicated GMK of the concept image of nonexamples. In mathematics, when considering the concept image of a nonexample, it should also come from the same domain as the example, in this case two-dimensional geometrical figures.

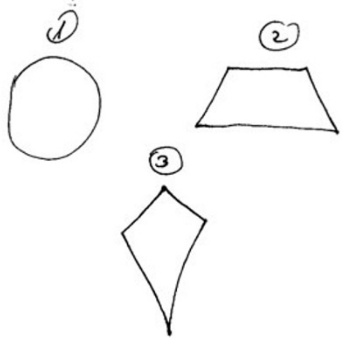



*Figure 3: One teacher's response to the task of drawing three nonexamples of triangles.*

*Assessing and building teachers' TMK and GMK regarding concept definitions*

During the same session described above, teachers were asked to write down on a piece of paper the definition of a triangle. Although no two definitions were exactly alike, each teacher was able to give a valid definition of a triangle. The following are some examples of definitions that the teachers wrote:

A triangle is a shape with three sides and three vertices.

A triangle is a polygon with three vertices.

A triangle is a geometric shape with three straight lines and closed.

A triangle has three sides and three angles.

In other words, the teachers demonstrated what may be considered TMK of the concept definition. Building teachers' GMK regarding concept defintions was done gradually and began by comparing the different definitions teachers had given for a triangle. The instructor gave the following instructions:

> Look at the definitions (now written on the board) and try to think which are correct and which are incorrect … if there are definitions which are unacceptable, explain why. If there are definitions which you approve of more than others, explain why. If there are definitions for which a slight revision may improve the acceptability of that definition, then write it. Perhaps there is more than one correct definition.

The teachers engaged in the task and then discussed the results together. Pointing to the first definition, "A triangle is a shape with three sides and three vertices", the instructor requested the teachers to raise their hands if they agreed that it was a valid definition. The following discussion ensued:

I: The question is very simple. Is this a definition of a triangle or not? That means that you can only vote yes or no. There is nothing in between and everyone has to vote.

H: How many times can I vote (yes)?

I: For each of the definitions you can either vote yes or no.

E: Is the question then if it's (the definition written on the board) closer to yes being a definition or closer to not being a definition?

I: There is no approximation. In mathematics it either is or is not (a valid definition).

In the above segment teachers come to realize that mathematical definitions must be precise. This is not only true of definitions for triangles but for all definitions of mathematical concepts. On the other hand, different definitions may be equivalent and thus there may be more than one definition for a particular concept. Although the instructor's approach may be considered quite direct, it became the norm with these preschool teachers that the instructor gave the closing argument of each discussion. Discussing the merits of each of the definitions led to a more general discussion of definitions:

E: Maybe we first need to know what a definition is.

R: A definition must be clear.

Y: That you don't argue with.

R: In a dictionary.

The teachers have begun to realize that it is important to first ascertain what is meant by a definition in mathematics before they can discuss if what is written may be considered a valid definition of a triangle. Differentiating between everyday dictionary definitions and mathematical definitions is another aspect of GMK related to concept definitions and was discussed further in the following lesson as teachers reviewed various definitions for a triangle found in dictionaries and mathematics textbooks.

During the next lesson Tall and Vinner's CICD theory was presented explicitly to the teachers. The teachers had been discussing which of the dictionary definitions would be unacceptable and for what reasons. One teacher quoted the following definition for a triangle, "a closed figure made up of straight lines." Another teacher responds, "But that can be like … a crown that you make. It doesn't say how many sides." This exchange prompted the instructor to introduce the notions of concept image and concept definition:

> Notice the connection between your thoughts and your knowledge, between your imagination and your knowledge … Vinner investigated mathematical concepts that also have a visual presentation. However, he also said that in mathematics we have definitions and we must work according to these definitions. This is the concept definition. The concept image is what we imagine in our thoughts when we close our eyes and think of the concept.

In the elementary school, concept definitions may be used to differentiate between critical and non-critical attributes of a concept, in order to identify examples and nonexamples of that concept. After introducing the notion of a concept definition, the instructor adds, "to define is to simply characterize a group of mathematical entities… to say what can be called by this (concept) name and what cannot." The instructor then refers to the examples and nonexamples of triangles the teachers drew during the first lesson pointing out that these illustrate each teacher's concept image whereas currently, the discussion at hand has revolved around the concept definition of a triangle.

In the above segments, the combined framework was essentially used by the instructor to assess current knowledge and then to direct and focus the knowledge being built. "From a cognitive point of view, prior knowledge has to be considered as a possibly influential characteristic" (Blömeke, Felbrich, Müller, Kaiser, & Lehmann, 2008). Assessing current knowledge is an essential first step to building new knowledge. The combined framework served to differentiate between TMK and GMK of the concept image as well as teachers' TMK and GMK of the concept definition. After assessing current knowledge the instructor began by focusing on GMK related to concept definitions leading eventually to an explicit discussion of the CICD theory.

### *Differentiating between subject matter knowledge (TMK and GMK) and KCT*

Throughout the program a clear differentiation was made between mathematical knowledge for the teachers and mathematical knowledge as it is applied in the classroom. Initially, teachers found it difficult to separate these two domains of knowledge.

> M: This is very confusing. You started off by talking about preschool children (in the beginning of the lesson) and now you decided to talk about mathematical thinking.
>
> I: Let's put things in order. First, we must talk about the mathematics as is. First we (the teachers) need to know what a triangle is. The children will wait. Tomorrow morning we are not going to talk with the children about triangles.
>
> A: I see us as preschool teachers, sitting with the students with the classic square, the classic rectangle, and the classic triangle and then we say, "What is this?" The child should say it's a triangle but according to what does he decide if it's a triangle or not?

> I: Just a second. We'll get there. We'll definitely talk about it but for now it's just us. Differentiating between the children and us is very important. Part of what we will learn will be important mathematical knowledge that we will know but that we won't necessarily tell it as such to the children because it may not be appropriate.

The instructor is stressing the difference between KCT and GMK and that they are two different ways of knowing mathematics. She further explains the necessity for this differentiation, "My strong belief is that first you need to know what you are dealing with mathematically because otherwise there will be no basis for how you answer the child."

During the second lesson, as teachers discuss various definitions for a triangle, the difference between GMK and KCT is again brought up:

> I: What is the source of this definition?
>
> C: A geometry text book.
>
> I: For which grade?
>
> M: Junior high school.
>
> Y: And you also need to know for what (mathematics) level the textbook is geared to.
>
> I: Ok. I want to make something clear. In the end, we will bring to the class a definition which we feel is appropriate for the preschool. But, now we are talking about definitions which would be acceptable to mathematicians … Now, you need to decide which definition is valid and which is not.
>
> H: Wait a minute. Are you talking about for us or for the children?
>
> I: For you.

The teachers are beginning to realize that a formal concept definition must be accepted by the mathematical community. This is part of the GMK being developed during these first two lessons related to concept definitions. On the other hand, knowing how to adapt a formal concept definition to the age and level of the students is an aspect of KCT. Although a triangle may be defined as a three-sided polygon, the teachers agreed that this definition would be unsuitable for young children for two reasons. First, it is quite unlikely that young children would comprehend the meaning of the term polygon. Second, a minimalist definition, although mathematically acceptable, does not stress all of the critical attributes that all examples share. As the instructor summarized:

> On the one hand, a definition in the preschool should take into consideration all of the critical attributes that are derived from the mathematical definition. On the other hand, it should take into consideration psychological aspects. We created a definition that includes closure, pointed vertices, straight sides, and the number three. Children should work according to this definition.

It was agreed that in the classroom children would be presented with the following definition: A triangle is a closed figure with three straight lines and three pointed vertices.[2]

The above segment illustrates how the combined framework was used to focus teachers' attention on the types of knowledge being built. In our program we found that teachers were eager to implement their newly acquired knowledge in the classroom. While this is, of course, commendable, the teachers needed to sort out the difference between the mathematical knowledge needed for teaching and the pedagogical knowledge needed to convey the mathematics to their students. By making this difference explicit, teachers were first able to focus on their knowledge of concept definitions and then focus on the teaching of concept definitions.

In the following section we describe how the combined theories tool was used to develop practitioners' KCS related to the concept image of triangles.

### *Building practitioners' KCS regarding concept images of triangles*

An important aspect of any teachers' knowledge for teaching is knowledge of content and students. Practitioners, who may not have been exposed to theories of children's geometric thinking, may still have acquired bits and pieces of this knowledge from their experiences taking care of children and engaging them with geometric activities (e.g. fitting shapes into puzzles). Yet, the practitioners may not be aware that they possess this knowledge or be aware that this knowledge is important. As such, the first step in enhancing this aspect of teachers' knowledge for practitioners was to elicit from them what they know of children's concept image regarding triangles. This took place in the very first session with the practitioners.

> I: We thought you could tell us a little about what you do in class regarding mathematics, shapes, and numbers.
>
> (A few of the practitioners give some examples of number activities in which they engage the children.)
>
> I: And what about shapes?
>
> L: They already know shapes.

From this short segment we get the impression that the practitioners engage their children with more number than shape activities and that they believe that the children already know shapes. However, we do not know what L means when she says that the children "know shapes". To draw the teachers into a more in depth discussion of what they know regarding children's knowledge of triangles, the instructor continues:

---

[2] It is important to note that precise language was used with the teachers as well as with the children. Terms such as corners and turns were not used. As such, "vertices" is the appropriate translation from Hebrew.

I: In most of the games, if children see a triangle, it's drawn like this one (points to the prototypical triangle, see Figure 4a). If they see this one (points to the scalene triangle, see Figure 4b.), will the children say this is a triangle?

G: [No] because children only recognize this (pointing the prototypical triangle), the regular triangle.

T: Or if it's (pointing to the prototypical triangle) upside-down or sideways, they will also understand that it's a triangle.

C: They'll say that this (pointing to the scalene triangle) is a line. I have a question. Can the point be on the bottom? Is that Ok?

G: It doesn't matter.

C: The children will say it's a triangle in every situation (orientation).

I: No, for the children it is definitely more difficult like this (holding the prototypical triangle uside-down – see Figure 4c). It's true that it has 3 points but the children will say that it's not a triangle because it's upside-down.

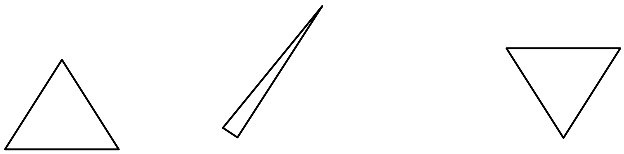


*Figure 4a* *Figure 4b* *Figure 4c*

*Figures 4a, 4b, and 4c: Prototypical triangle, scalene triangle, and prototypical triangle turned upside-down.*

What do the practitioners know regarding the children's concept image of triangles? First, they seem to agree that the prototypical triangle with a horizontal base will be easily recognized as a triangle. They also seem to be in agreement that the scalene triangle will not necessarily be part of children's concept image of triangle. Questions arise when discussing the upside-down triangle. First, C is not herself sure that the upside-down triangle is a triangle. On the other hand, she believes that children will call it a triangle. T also believes that children will readily agree that the prototypical triangle, in any orientation, is a triangle. As research (Hannibal, 1999) has shown orientation to be a hindrance, the instructor at this point intervenes in order to correct the practitioners' perception of children's concept image. The instructor then moves on to children's concept image of non-triangles.

I: They'll know that this (see Figure 5a) is not a triangle. But this (see Figure 5b) …

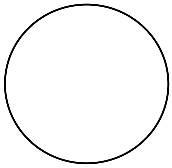


*Figure 5a*

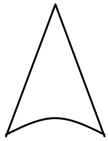


*Figure 5b*

*Figures 5a and 5b: A circle and a "triangle" with curved side.*

C: They'll say that it's a clown hat.

G: Or a kite.

L: I would also say that it's a clown hat.

I: But the question is, will they say that it's a triangle or that it isn't a triangle?

H: They see the sides, that there are 3 sides.

L: They don't look to see if the line is straight or curved.

H: In my opinion, they'll say that it's a triangle because it has 3 sides. They won't notice the curve.

I: Exactly. They won't understand that the side of triangle must be straight. The truth is, with such young children, we don't really know. We have to check and see what will happen.

The above segments illustrate how the instructors elicited from the practitioners their knowledge regarding children's concept image of both examples and nonexamples of triangles. Most important is the instructor's acknowledgement in the last statement of the need for further investigation. At the end of the first session the instructor is more direct and assigns homework, "Ask children: Is this a triangle? Why?" In other words, in order to enhance practitioners' and teachers' knowledge of students, it is important to know what to ask children and what to listen for.

The results of the homework assignment were discussed in the next session. Some of the practitioners, especially those in charge of caring for the very young children, carried out the assignment with their own children or grandchildren. Others carried out the assignment with the three-year old children in the daycare center.

G: I have a child two years and 8 months old. I drew a picture of a regular triangle (meaning a prototypical triangle) and asked him, what is this? He said it was a clown hat. I drew a regular triangle.

E: I have a five year old grandchild. I drew for her a triangle with a curved side. She said it was a triangle.

H: I work with the infants and don't have any small children or grandchildren at home.

C: I sat with a group of [three-year old] children (in the day care center). And I asked them, what is this? And then I asked them, why? (See Figure 6.) (C shows the instructor and the other practitioners the figures she drew for this activity. Note that C drew five examples of triangles and one nonexample. Out of the five examples, only one did not have a horizontal base. Triangle 5 was intended to be a narrow triangle but the lines merged on top.) This one (pointing to shape 5) was a little difficult. This one (pointing to shape 6), Yarden said was a triangle because it's like the Star of David. And this one, (pointing to shape 2) they thought was a triangle! There were really children who didn't know [that it wasn't a triangle]! So, I showed them that you have to draw a straight line from point to point in order for it to be a triangle.

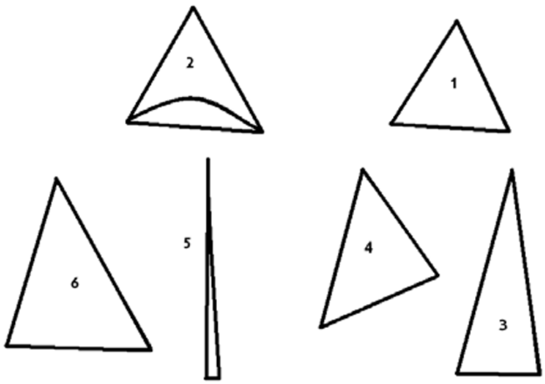



*Figure 6. Six figures C drew and used in an activity with three-year olds.*

From the reports above we see that most practitioners gained first-hand experience into investigating children's concept image of triangles. At this point, the example space of the practitioners was still narrow. However, as the program continued, the practitioners were exposed to additional examples and nonexamples of triangles.

Part of KCS is knowing how children's concept image develops. This was discussed periodically during the professional development course with the practitioners. Already during the first session, the instructor pointed out that even three-year old children bring to class knowledge they have gained from their homes and environments, "They (the children) have begun to build for themselves a world of images – this is [a triangle] and this isn't [a triangle] … For example, children will say this (pointing to the prototypical triangle) is a triangle because it looks like the roof of a house." It is also important to know that concept images may sometimes include misconceptions. Referring back to the scalene triangle that the practitioners knew would be difficult for children to accept as a triangle, the instructor also points out, "Children will say that this [scalene triangle] is not a triangle because it doesn't resemble the roof of a house."

During the second session, after practitioners have become accustomed to discussing children' knowledge of shapes, the instructor spends more time on why it is important to develop children's concept image of triangles at such a young age:

> Children absorb from their surroundings, [what is a] circle [and what is a] triangle. So, at least they should learn correctly. You know, if you learn something incorrectly, afterwards it's much more difficult to correct the misconception. (The practitioners nod in agreement.) Therefore, we try as much as possible to teach correctly. And what does that mean?... Among other things, [it means] to present many examples of a triangle, all sorts of triangles. Ok? …Think about a boy who grew up in a place which had only one small white dog, a poodle dog. That's all he knows about dogs. Now think what would happen if he saw an amstaff dog. He for sure would not think it was a dog…What do children see? They see all the time nice triangles because in the puzzles the triangles are always nice, with equal sides. So, that's what a child learns because that's what he sees.

The instructor ends by emphasizing that it is necessary to present and expose young children to many different types of triangles in order to widen children's concept image of triangles. The rest of the session is spent on discussing various examples and nonexamples of triangles which leads into knowledge of teaching triangles.

### *Building preschool teachers' KCT regarding concept definitions and concept images of triangles*

The formation of geometrical concepts, as with many mathematical concepts, is a complex process in which examples play an important role (Watson & Mason, 2005). Initially, the mental construct of a concept includes mostly visual images based on perceptual similarities of examples, also known as characteristic features (Smith, Shoben, & Rips, 1974). This initial discrimination may lead to only partial concept acquisition. Later on, examples serve as a basis for both perceptible and nonperceptible attributes, ultimately leading to a concept based on its defining features. Visual representations, impressions and experiences make up the initial concept image. Formal mathematical definitions are usually added at a later stage. According to the Principles and Standards for School Mathematics (NCTM, 2000), young children "need to see many examples of shapes that correspond to the same geometrical concept as well as a variety of shapes that are nonexamples of the concept" (p. 98). Thus, another important aspect of KCT is knowing which examples and nonexamples to present to children that will promote the development of an appropriate concept image as well as encourage children to refer to the concept definition.

In this section we describe a segment which took place with a second group of preschool teachers during the fifth lesson of their course. The teachers had been instructed to assess their children's knowledge regarding the identification of examples and nonexamples of triangles and are now discussing the results. It soon becomes obvious that the results were largely dependent on the choice of examples and nonexamples the teachers had chosen to use for this assessment. (See Figure 7 for a sample of some of these examples and nonexamples). The instructor explains:

> The results do not give us a complete picture of what the children know and what they are capable of knowing. We have found in our work with children that almost all of the children correctly identify this (pointing to an equilateral triangle with a horizontal base) as a triangle and only a third of the children will correctly identify the same triangle if it is turned upside down. The typical concept image of the triangle is this (pointing to an equilateral triangle with a horizontal base).

In other words, in order to properly assess children's knowledge, the teacher should include examples that are not necessarily part of the child's intuitive concept image.

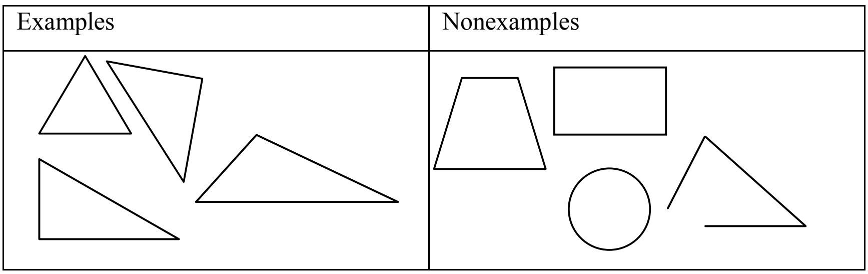



*Figure 7. A sample of examples and nonexamples of triangles used by teachers.*

Choosing examples that are not necessarily part of the child's concept image may also encourage the child to refer back to the concept definition (Tsamir, Tirosh, & Levenson, 2008). As the instructor claims, "it is important to work with many examples and nonexamples… going over the critical attributes and at the same time creating a world of images." The teachers are then instructed to think about the figures along two dimensions: a mathematical dimension and a psycho-didactical dimension. The mathematical dimension divides the figures into examples and nonexamples of triangles according to the concept definition. The psycho-didactical dimension divides the figures into what is and is not intuitively identified as examples and nonexamples according to the child's current concept image.

Knowledge of how to choose appropriate examples and nonexamples was evident later on during the course as teachers discussed pentagons. In order to create the examples, teachers discussed the concept definition of a pentagon:

S: I want to know if there is an exact definition for a pentagon.

R: A closed figure with five sides.

O: A five-sided polygon.

S: Ok.

I: And what about a definition for the children?

S: For the children I would say five sides, five vertices, and closed.

O: A closed figure… like we did before… with five sides and five vertices.

Working together in groups, the teachers came up with the following suggestion of examples and nonexamples to use in various activities (see Figure 8).

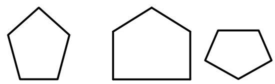
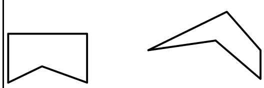
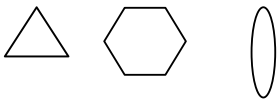
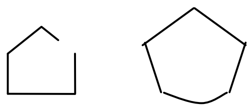
| Dimensions | Psycho-didactical | |
|---|---|---|
| Mathematical | Intuitive | Non-intuitive |
| Examples |  |  |
| Non-examples |  |  |

*Figure 8. Teachers' suggestions of examples and nonexamples of pentagons.*

It may be surprising that the teachers placed the upside down pentagon in the section for intuitive pentagons. After all, the teachers had previously experienced that upside down triangles are not necessarily part of the child's concept image of a triangle. However, at this point, the teachers felt that the upside down pentagon may be considered intuitive. The children had already been presented with triangles of various orientations and could successfully identify an upside down triangle as an example of a triangle. In other words, the children's concept image of triangles had changed and the teachers were choosing examples based on the children's current concept image of geometrical figures. On the other hand, triangles cannot be concave and so concave figures, such as the concave pentagon, may not currently be part of the child's concept image. The teachers had gained knowledge of their students (KCS) and used this knowledge in their teaching (KCT).

The relationship between knowledge of students and knowledge of teaching was observed several times during the year. Towards the end of the year, the teachers discussed how to help children who still had difficulties identifying various examples and nonexamples of geometrical figures. Referring to triangles, one teacher stressed the need to help children recall the concept definition. She suggested the following:

> E: First we need to strengthen the critical attributes. So, we start with the triangle they are used to (referring to the equilateral triangle) and we put it down in different directions and ask the child what has changed and what has not and to check again the critical attributes. Regarding the hostile triangles (referring to those which do not coincide with the child's concept image) I would greatly enlarge the triangle so it would be much clearer to the child and ask him again to check the critical attributes, the sides and vertices.

Notice that this teacher has identified two possible stumbling blocks for the children. The first is children's difficulty with orientation. She isolates this difficulty using the triangle most likely to coincide with children's concept image and focusing only on the changing orientation. The second is children's difficulty in identifying non-intuitive triangles. Her suggestion of enlarging the triangles directs the child to notice the straight sides and pointed vertices of the triangle. Similarly, when discussing children's difficulties in identifying concave pentagons, a different teacher suggests enlarging the figure and cutting it out so that children can feel the hidden vertex. During the next lesson, this teacher described how she carried out this suggestion and that the enlarged concave pentagon was indeed helpful.

In this segment we see the results of explicitly introducing preschool teachers to the CICD theory and explicitly discussing with them the difference between the mathematical knowledge they as teachers need to know and applying this knowledge in the preschool. We can see a clear difference between the examples and nonexamples teachers chose for triangles in the beginning of the year to those they chose for pentagons in the middle of the year. Teachers are cognizant of the need to present a suitable definition of a pentagon for their children. They are aware of the tension between the concept image and concept definition and devise activities that will enrich the children's concept image while strengthening their awareness of the concept definition. Using the combined theory as a lens we may say that the teachers are accessing their KCS related to concept images and concept definitions in order to build their KCT related to concept images and concept definitions.

*Summing up and looking ahead*

In this chapter we illustrated how a theoretical framework which combined a theory of teachers' knowledge with a theory of mathematical knowledge may be used as a tool in promoting teachers' mathematical knowledge for teaching. "A crucial trait of a valuable framework of teacher knowledge is the extent to which it identifies that knowledge needed for student learning and understanding" (Graeber & Tirosh, 2008, p. 124). Other tools conceptualize teachers' knowledge based solely on the work teachers do. We add that it is equally important to frame teachers' knowledge based on the knowledge we wish our students to gain. Viewing teachers' mathematics knowledge through two lenses, one of teachers' knowledge and one of mathematical knowledge, allows us to pinpoint more precisely what teachers need to know for teaching mathematics. Of course teachers need to know which examples of triangles to present and in what order to present them (KCT). However, if we recognize that some examples will enhance students' concept image of a triangle and others will encourage students' use of the definition we may accordingly develop teachers' mathematical knowledge for teaching each of these aspects.

The theorized tool we described combined aspects of Ball and her colleagues' conceptualization of teachers' knowledge for teaching with Tall and Vinner's CICD theory in order to promote preschool teachers' knowledge for teaching geometry. There are several variables in the implementation of the tool. There is the mathematical context, the grade-level at which the teachers taught, the action taken with the tool, and the theories we chose to combine. Each of these variables represents possible directions for further development and wider use of the tool in enhancing teacher's knowledge for teaching mathematics.

Regarding the mathematical context, we found that for preschool teachers, the context of geometry provided a natural venue for discussing images and definitions. Beginning with triangles and other two-dimensional polygons, the teachers could readily discuss the figures they saw and drew and began to understand the need for concept definitions. They also came to acknowledge that not every concept definition may be adapted for the young children in preschool. This came up when discussing circles and the concept image and concept definition of a circle. It was decided that for the circle, a child's concept image may currently be enough. These discussions carried on as the teachers discussed three-dimensional solids such as pyramids, spheres, and cylinders. Although this book specifically focuses on the context of geometry, we believe that the generality of the CICD theory allows it to be applied to building teacher's knowledge of additional mathematical contexts. In the preschool, for example, we used the combined framework for building teachers' knowledge of equivalent sets (Tirosh, Tsamir, Levenson, & Tabach, 2011). As with geometry, we used the combined theory to build teachers' TMK of the concept image of equivalent sets and differentiated between this knowledge and KCT regarding this concept image. The same was done for the concept definition. If the use of this tool is to be expanded to other preschool mathematical contexts (such as patterns and measurement), then perhaps prior research will be necessary in order to first investigate children's concept image and concept definition in these contexts.

Regarding the grade-level at which the teachers' taught, this chapter illustrated promoting knowledge for teaching children ages three till six. We believe that the combined theory has potential to be used as a tool for promoting teachers' knowledge for teaching in other grades as well. In both elementary and secondary schools, studies have shown that tension exists between students' concept images and concept definitions within various mathematical contexts (Bingolbali & Monaghan, 2008; Gray, Pinto, Pitta, & Tall, 1999; Even & Tirosh, 1995; Schwarz & Hershkowitz, 1999; Levenson, Tsamir, & Tirosh, 2007; Vinner & Dreyfus, 1989). Perhaps at the high school level, teachers are more cognizant of the necessity for definitions than preschool teachers are. On the other hand, they may pay less attention to concept images. This issue will need to be addressed by perhaps placing extra emphasis on these cells during professional development.

Another issue that arises from pondering the use of the tool in professional development is the degree of explicitness when presenting the tool to teachers. Upon reflection, the four dimensions of teachers' knowledge were not made as explicit to the teachers and practitioners as was the concept image-concept definition theory. We believe that it is important to make both theories equally explicit to teachers. We also believe that it is important to first build teachers' TMK and GMK and then build on this knowledge when developing KCS and KCT. This issue is being addressed in our current courses where the four dimensions of teachers' knowledge are explicitly presented and discussed.

Choosing which theories to combine is a significant issue which needs to be addressed. Regarding our goals for professional development, it is too simplistic to say that we aimed to enhance teachers' knowledge. As the previous chapter indicates, conceptualizing mathematical knowledge for teaching is complex. Our choice of breaking down Shulman's (1986) notion of SMK into TMK and GMK arose from the necessity to not only enhance teachers' knowledge of geometry but to connect this specific knowledge to the larger issue of knowing mathematics in general. Choosing to use Ball and her colleagues' notions of KCS and KCT also arose from our necessity to use a finer grain tool than provided by Shulman's (1986) often used notion of PCK. In retrospect, we found that these choices were well suited for conceptualizing the knowledge needed for teaching geometrical concepts in preschool and that it is important to pay attention to each of the four domains of knowledge. Familiarizing preschool teachers with the concept image concept definition theory may enlighten teachers to the tension which may exist between the concept image and concept definition and inform their instruction in ways that will promote children's advancement along the van Hiele levels of thinking.

The first chapter in this part of the book was quite theoretical in that it presented background, position papers, theories, and research regarding preschool teachers' knowledge for teaching geometry. The second chapter illustrated how theory may be put into practice. In this last chapter we immerse ourselves in the practice of professional development by focusing on the tasks that we used in our professional development course.

CHAPTER 9

# TASKS IN THE PROFESSIONAL DEVELOPMENT OF PRESCHOOL TEACHERS

In Part Two of this book, we discussed at length different aspects of tasks that ought to be considered when designing and implementing geometrical tasks with children. Many of those aspects may be applied to tasks that are intended to be implemented with both prospective and practicing preschool teachers. In addition to those issues raised previously, when considering tasks for teachers, one may also consider the purposes for specifically having teachers engage in tasks. Watson and Sullivan (2008) listed four purposes:

> Purpose 1. To inform them about the range and purpose of possible classroom tasks.
>
> Purpose 2. To provide opportunities to learn about mathematics.
>
> Purpose 3. To provide insight into the nature of mathematical activity.
>
> Purpose 4. To stimulate and inform teachers' theorising about students' learning. (p. 110)

In their chapter on tasks for teachers, Watson and Sullivan (2008) go on to discuss in depth different types of tasks that may be used with teachers, as well as the affordances and constraints of each. Tasks may also be examined in light of different theories such as the combined-theories framework we outlined and illustrated in the previous two chapters.

It is not the intention of this chapter to review at this point additional theories related to tasks and teachers. Instead, in this chapter, we present six geometry-related tasks that we used in our professional development courses for preschool teachers.[1] Some of these tasks were mentioned or related to in different sections throughout this book. Yet, in general, the tasks were not presented in their entirety. In this chapter, we gather together the different tasks as well as present some additional tasks in order to offer a practical perspective. Thus, for each task, we present, in a succinct matter, the aims of the task, as well as actual handouts or activity sheets and directions for use.

[1] Research related to professional development for kindergarten teachers was supported by the Israel Science Foundation, Grant Noo. 654/10.

*Task 1:* Drawing examples and nonexamples

*Aims:*

1. To investigate teachers' concept image of examples and nonexamples of a specific geometrical concept.
2. To introduce the notions of concept image and example space to teachers.

*Handouts:*

Only a blank sheet is necessary.

*Instructions:*

The instructor hands out a blank sheet of paper and says, "Draw a triangle." After making sure that each teacher draws one triangle the instructor should then say, "Now, draw another triangle." After each teacher draws a second figure, the instructor should say once more, "Now, draw another triangle." The instructor then requests the teachers to turn over the sheet of paper and gives the following instruction, "Draw a figure which is not a triangle." After each teacher draws a figure, the instructor should say once more, "Now, draw another figure which is not a triangle." After each teacher draws a second figure, the instructor should say, "Now, draw another figure which is not a triangle."

Note: The name of any two-dimensional figure may be used in the above task in order to investigate and then discuss teachers' concept image of examples and nonexamples of that concept.

*Task two: Hierarchical relationships between figures*

*Aims:*

To investigate teachers' knowledge of the hierarchical relationship between quadrilaterals, rectangles, and squares.

*Handouts* (See Appendix A – Task two)

*Instructions*:

Each teacher is given a copy of the handout. The instructor says, "Look at each of the figures on the first handout. Is the figure a quadrilateral? A rectangle? A square? None of the above? On the second handout place a check mark if the name of the column is appropriate for the figure. Place an X if it is not appropriate."

Note: This task may be used either to investigate teachers' knowledge before discussing with them the hierarchical relationship between quadrilaterals, rectangles, and squares or it may be used to summarize a lesson which discussed these relationships.

*Task 3: Sorting figures*

*Aims:*

1. To raise awareness among teachers to the variety of examples and nonexamples which may be presented to children.
2. To raise teachers' awareness of the psycho-didactical dimension as well as the mathematical dimension of geometrical figures and to offer teachers a way to categorize the vast amount of examples and nonexamples along these dimensions.

*Handout with instructions* (See Appendix B):

Note: This task may be implemented after discussing with teachers the difference between visual and attribute reasoning or as a prelude to that discussion. It may also be used as a base for discussing how to choose which examples and nonexamples to present to the children at different points in their learning. After implementing this task with teachers for the first time, it may then be used in order to help teachers consider the examples and nonexamples which they may present for other two-dimensional as well as three-dimensional geometric figures.

*Task 4: Responding to children*

*Aims:*

1. To investigate how teachers would respond to children's correct and incorrect identifications of examples and nonexamples of different figures.
2. To investigate how teachers would respond to children's explanations which accompany correct and incorrect identifications of different figures.
3. To spark a discussion regarding the difference between explanations which are based on visual reasoning and explanations which are based on attributes.
4. To spark a discussion of how teachers may assess children's geometrical reasoning.

*Handout with instructions*: (See Appendix C)

Note: This task has several variables which the instructor should consider. First, it is highly dependent on the figures presented on the handout. In the above handout we chose one intuitive example of a pentagon (the first figure) and one non-intuitive example (the concave pentagon). Regarding the nonexamples presented in this particular handout, each illustrates a breach of one critical attribute. The second important variable relates to the children's explanations given as examples on the handout. On this handout there are explanations based on visual reasoning (it looks like a pentagon) and those based on critical attributes. Even when a child bases an explanation on a critical attribute, it may be either wrong (claiming that the concave pentagon has four points) or insufficient (claiming that the pentagon has five lines).

*Task 5: Knowledge of children's ways of thinking*

*Aims:*

1. To investigate teachers' knowledge of cylinders.
2. To investigate teachers' knowledge of children's ability to identify cylinders.
3. To investigate teachers' knowledge of possible errors that children make when identifying cylinders.

*Handout with instructions written* (see Appendix D)

Note: As with the previous tasks, the instructor may vary the examples and nonexamples presented on the handout.

*Task 6: Describing three-dimensional geometric figures*

*Aims:*

1. To investigate three-dimensional geometrical figures
2. To identify the critical attributes of three-dimensional geometrical figures.
3. To bring to light the relationships between two and three-dimensional figures.

*Instructions*:

The instructor places a variety of three-dimensional geometrical figures on a table such as cylinders of various heights and diameters, cones of various heights and diameters, spheres, pyramids, cubes, prisms, etc. The teachers are encouraged to not only look at the figures but to pick them up, feel them, and experiment with their movement. The instructor then gives each teacher the handout.

*Handout:*

Below you will find a bank of words. Use these words to describe each of the figures on the table. Describe what you see as well as what you feel.

*Word bank:*

Vertex, edge, face, height, base, circle, triangle, pentagon, square, rectangle, unfold, surface, rolls, flat, curved, cross-section, cylinder, prism, sphere, pyramid, cuboids.

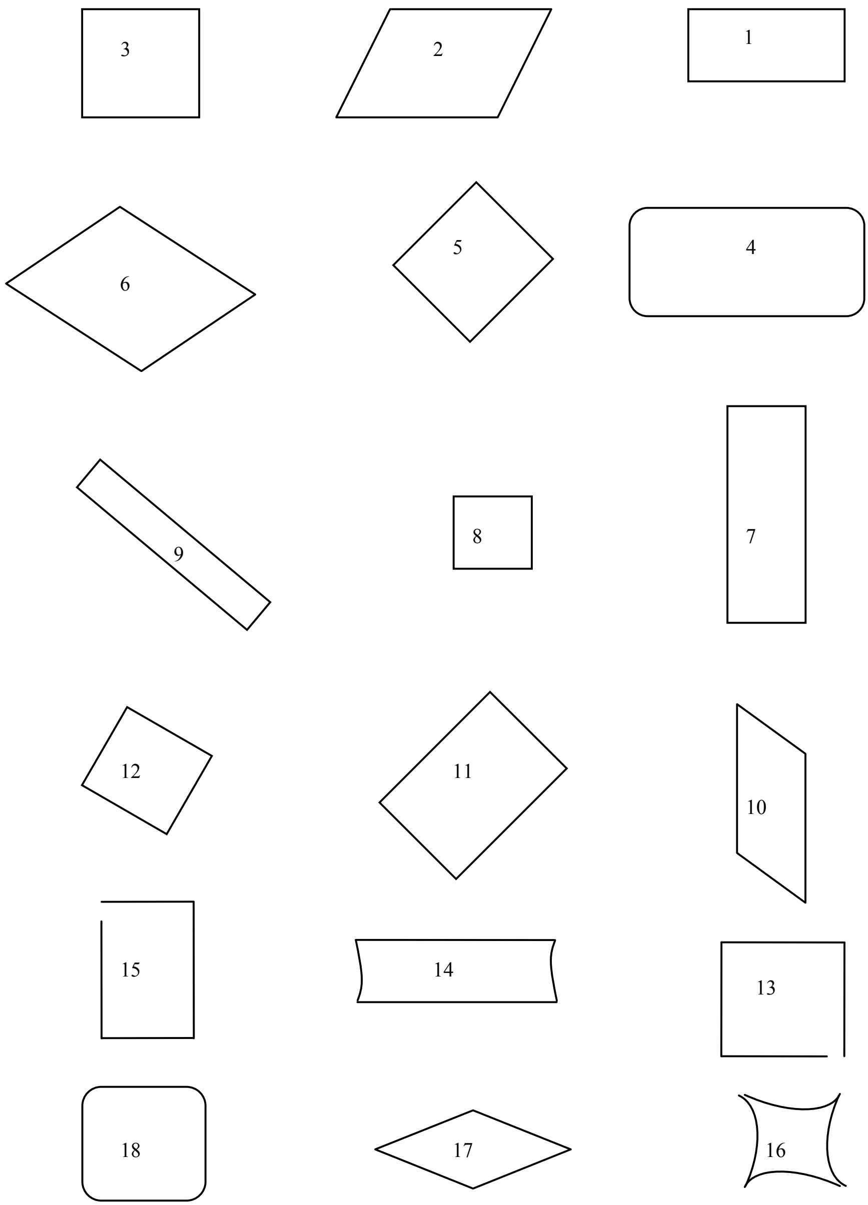
Note: Because this task deals with solids, it is important for the teachers to feel the difference between the figures and not just observe them from a distance. For example, cones and cylinders feel different even when your eyes are shut. You can see that a cone has one circular base while the cylinder has two circular bases. You can feel that a cone has one vertex because you can feel the point. You can feel that the cylinder does not have any points. You can give a nudge to the cylinder and then observe how it can roll back and forth. When you push the sphere it may roll in all directions. In addition, the word bank focuses the teachers on word usage, providing an opportunity to discuss the differences between every-day meanings and mathematical meanings of the same word.

## SUMMARY

The tasks described above may be used to both investigate as well as promote preschool teachers' knowledge of geometry. In essence, throughout professional development we continuously do both. We assess what the teachers know and then build on this knowledge. As noted above, tasks in professional development serve various purposes (Watson & Sullivan, 2008). The tasks we presented here, as well as many of the tasks presented throughout the book, first and foremost, provide opportunities for the teachers to learn geometry. Knowledge of geometry is always at the heart of a task. In addition, tasks may also serve as the basis for discussing children's knowledge and theories related to children's geometrical development. Finally, some of the tasks implemented with teachers, may be adapted for implementation with children. Many similarities exist between the tasks discussed in Parts One and Two and the tasks discussed in Part Three. Thus, implementing the tasks above with teachers may offer them ideas for tasks which they can then implement in their own preschool classes.

# APPENDIX A

*Task 2 – Handout A*



*Task 2 – Handout B*

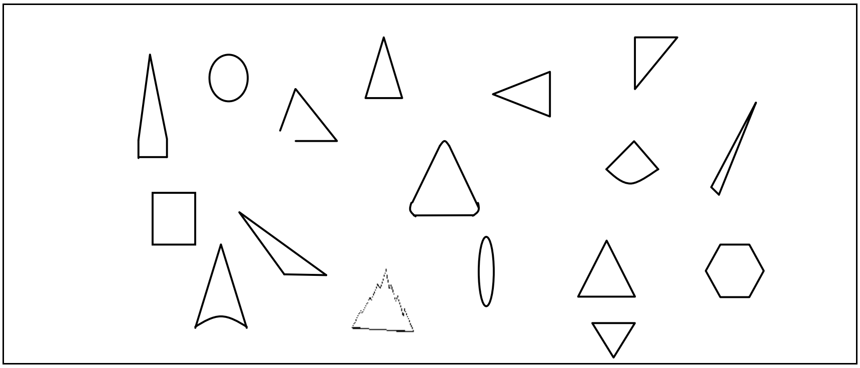
Each of the figures presented in Handout A are numbered 1 through 18. Place a check mark √ if the name of the column is appropriate for the figure. Place an X if it is not appropriate.

| | Quadrilateral | Rectangle | Square | Other |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |
| 7 | | | | |
| 8 | | | | |
| 9 | | | | |
| 10 | | | | |
| 11 | | | | |
| 12 | | | | |
| 13 | | | | |
| 14 | | | | |
| 15 | | | | |
| 16 | | | | |
| 17 | | | | |
| 18 | | | | |

## APPENDIX B – HANDOUT FOR TASK 3

Below are many different figures. Sort the figures into examples and nonexamples of triangles. Then sort the examples into those which preschool children would readily identify as triangles and those which they children may have difficulty identifying as triangles. Now sort the nonexamples into those figure which preschool children would readily identify as not being triangles and those which the children may have difficulty identifying as not being triangles. Copy your four groups of figures onto the space provided. After you have sorted all of the figures below, add additional figures to each category.

Figures to be sorted:

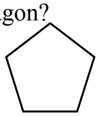
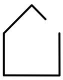
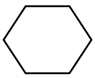
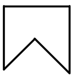


**Sort the above figures into the following four categories and then add some figures of your own to each category.**

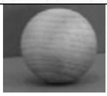
| Triangles | Examples | Nonexamples |
|---|---|---|
| Easily identified by children | | |
| Children have difficulty identifying these figures | | |

## APPENDIX C – HANDOUT FOR TASK 4

Danny is a child at the end of his kindergarten year. This page presents questions that were posed to Danny, as well as Danny's responses to those questions. Fill in the blank spaces left for the kindergarten teacher's comments.

| The question that was asked to Danny | Danny's response | Comments (Was Danny correct)? | Danny's explanation | Comments (Would you accept Danny's explanations?) |
|---|---|---|---|---|
| Is this a pentagon?  | Yes | | Because it has five lines. | |
| Is this a pentagon?  | No | | Because it's not closed. | |
| Is this a pentagon?  | Yes. | | Because it looks like a pentagon. | |
| Is this a pentagon?  | No. | | Because it has four points. | |

# APPENDIX C – TASK 5

| The figure | Is it a cylinder? | Why? | Will kindergarten children correctly identify this figure as an example or a nonexample of a cylinder? | Will children be able explain their identifications? |
|---|---|---|---|---|
|  | | | | |
|  | | | | |
|  | | | | |
|  | | | | |
|  | | | | |
|  | | | | |

# EPILOGUE

This book has been about the odyssey, the intellectual wandering and eventful journey, of teaching geometry to preschool children. It began with a preliminary discussion of theories and research related to developing geometrical concepts and reasoning among young children, moved on to the tasks which may be used in this endeavor, and ended with the teachers and the knowledge and preparation they need for teaching geometry in preschool. We call this an odyssey because as one wanders through the book, one finds many events which call upon the reader to linger a bit longer, to contemplate a particular example, illustration, or situation. What would you do or what have you done in similar situations? How may a certain theory or task be implemented in your preschool classroom or in your professional development program or in your research study?

A proper intellectual journey also gives rise to many dilemmas. In the first part of this book, we studied preschool children's development of geometrical concepts, placing great emphasis on the examples and nonexamples presented to children. Yet, some questions remain unanswered. Should we first present to children only examples, intuitive and non-intuitive, and only later present nonexamples? Perhaps we should first present intuitive examples and intuitive nonexamples and then, later on, present non-intuitive examples and non-intuitive nonexamples? Is there a hierarchy within the group of non-intuitive examples and within the group of non-intuitive nonexamples? In other words, even though many figures may be considered non-intuitive examples of a certain shape, are some a little easier to identify than others, and if so, what makes it so? Regarding specific shapes, should squares be presented as examples of quadrilaterals or examples of rectangles? Perhaps, as with circles, we should reinforce children's concept image of squares without relating to the relationships between squares, rectangles, and quadrilaterals.

In the second part of this book we focused on geometrical tasks to be implemented with children. Not all tasks are suitable for all children. Which tasks are more suitable for three-year olds and which may be implemented with five-year olds? Which tasks may be tailored for children with special needs? Taking into consideration the limited fine motor skills of young children, should we encourage children to draw shapes when it is more than likely that the shapes will not consist of straight lines and pointy vertices? In the second part we also saw an example of how children's engagement in geometrical tasks may promote monitoring behaviors. If this is the case we may ask ourselves, can promoting monitoring behaviors, and perhaps other control mechanisms essential for problem solving, be worked intentionally into the design of geometrical tasks for preschool children? In general, how can geometrical tasks be designed and implemented in order to promote other mathematical processes, such as problem solving, communication, and reasoning?

In the third part of this book we discussed professional development that aims to increase preschool teachers' knowledge for teaching geometry. Not all preschool practitioners and teachers have the same educational background. The question remains as to how professional development programs may be tailored to meet the different needs of different teachers. Another question that arose from this part of the book was related to the tools used in professional development. In our programs, we used theories and tasks as tools for promoting knowledge. However, other tools might also be considered. Sherin and Han (2004) investigated the use of video-clubs in promoting middle-school mathematics teachers' pedagogical content knowledge related to students' ways of thinking. Can such a tool be used with preschool teachers? Currently, we are investigating the use of video feedback as a tool for promoting preschool teachers' knowledge to teach geometry. In one of our professional development programs, preschool teachers are taping themselves in action as they engage children in geometrical tasks. How might such an intervention promote teachers' knowledge of children's difficulties? How might such an intervention affect teachers' knowledge of appropriate tasks?

The questions mentioned above are important as they represent what is still unknown, what requires further investigation. In addition, as with any journey, we also consider the paths not taken. For example, one issue not discussed in this book was how the child's home environment may play an active role in the development of geometrical concepts and reasoning. In some cultures, children stay home and do not begin any schooling until the age of six or seven. But, even in countries where children enter daycare at a very young age, parents and caregivers may have a significant role to play in the education of young children.

Affect is another issue which was not addressed in this book. Affective aspects of mathematics education include beliefs, attitudes, values, and emotions. These issues relate to both teachers and their young students. In a recent study we investigated kindergarten children's self-efficacy beliefs regarding their ability to complete geometrical tasks (Tsamir, Tirosh, Levenson, Tabach, & Barkai, 2011). Preliminary results suggested that kindergarten children believe greatly in their ability to identify some geometrical shapes. However, children's high self-efficacy does not necessarily correlate with their knowledge. Currently, we are investigating preschool teachers' self-efficacy beliefs regarding their ability to complete geometrical tasks as well as their self-efficacy beliefs regarding their ability to teach geometry in preschool.

This book focused on the development of geometric concepts and reasoning among young children. Although geometry is an important strand, it is not the only strand to be considered in children's mathematical development. How might children's development of geometric reasoning be related to the development of other mathematics concepts such as numbers, patterns, and measurement?

As a final note we would like to add that, for us, this journey has not ended. If we are to improve mathematics education for young children, then additional research is needed, research which takes into consideration both the children and their teachers. We are encouraged by the recent call from policy and position papers to improve preschool mathematics education. We are encouraged by the research which is currently being undertaken by fellow mathematics education researchers. We hope that this book has been of value to preschool teachers, preschool teacher educators, and preschool mathematics education researchers. We also hope that this book will serve as an impetus for further research.

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